2022 Edition

CALCULUS

Essential Skills Practice Workbook with Full Solutions

Derivatives, Limits and Integrals

$$\frac{d}{dx}\sqrt{1+\sin x} \qquad \frac{d}{dx}\frac{e^{-3x}}{1+e^{2x}} \qquad \text{Find } \frac{dy}{dx} \text{ if } y^5 - \sin x + y = 0$$

$$\frac{d^2}{dx^2}x^2e^x \qquad \int_{\theta=\pi/4}^{\pi/3} \cot\theta \; d\theta \qquad \int_{x=1}^{\infty} e^{-x}\,dx \qquad \int_{x=y}^{2y}\int_{y=0}^{z}\int_{z=0}^{3} \frac{xy}{z^2}\,dx\,dy\,dz$$

TOPICS COVERED: • Basic differentiation rules including power rule, chain rule, product rule and quotient rule • Derivatives of polynomial, trigonometric, inverse trigonometric, exponential, hyperbolic, logarithmic and implicit functions • Second order derivatives • Extreme values - maxima and minima • Limits and l'Hôpital's rule • Indefinite and definite integrals of polynomial, trigonometric, exponential, logarithmic and hyperbolic functions • Integration by polynomial and trigonometric substitutions • Integration by parts • Multiple integrals

Sudhir K. Sood, Ph.D.

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Essential Skills
Practice Workbook
with Full Solutions
Derivatives, Limits and Integrals

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Calculus: Essential Skills Practice Workbook with Full Solutions- Derivatives, Limits and Integrals 2022 Edition

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INTRODUCTION

This workbook has been written to provide all the help to master the calculus skills for evaluating the derivatives, limits and integrals quickly and accurately.

The subject matter of the workbook has been divided into twenty one chapters with each chapter dealing with a particular topic. Basic differentiation rules including the power rule, the addition rule, the chain rule, the product rule and the quotient rule are considered first to provide a conceptual framework for evaluating derivatives. Derivatives of trigonometric, inverse trigonometric, exponential, hyperbolic, logarithmic and implicit functions are discussed next. We then examine the idea of second derivatives before discussing how to apply the ideas learnt till now to find the maxima and minima of a function. The important concept of limits and their evaluation by various methods including the use of l'Hôpital's rule is considered next. Having studied derivatives and limits, we introduce the ideas of indefinite and definite integrals before learning how to find the integrals of polynomial, trigonometric, exponential, hyperbolic and logarithmic functions. The important methods of integration by polynomial and trigonometric substitutions and integration by parts are studied next. Lastly, we learn how to evaluate the multiple integrals.

Each chapter of the workbook begins with a concise yet self-contained discussion of the topic of the chapter supplemented by formulas, identities and properties of various functions, their derivatives and anti-derivatives. This is followed by a number of examples that are solved step by step with explanation of every step. The accompanying notes will help the reader recapitulate the calculation skills involving algebraic, trigonometric and other functions– including the 'simple' skills like addition of fractions, multiplication/division of exponents and so on.

At the end of every chapter, adequate and variety of practice exercises are provided to test your problem solving skills as well as your comprehension and retention of the topic. Fully explained solution of every practice exercise is provided at the end of every chapter to allow you to check your answers.

It is hoped that with all these feature, the workbook will assist students not only in learning the basic calculation skills of finding derivatives and integrals but will prove to be a convenient handbook of rules, formulas and methods which they will keep referring to during their future studies.

1 BASIC DIFFERENTIATION RULES AND THE DERIVATIVES OF POLYNOMIALS

Consider the term ax^n, where the coefficient a is a constant and the power *n* of variable x is a real number: zero, positive, negative or a fraction. To find its derivative with respect to x, first multiply a by n and then multiply by x raised to power $n - 1$ to get the following formula:

$$\frac{d}{dx}(ax^n) = anx^{n - 1}$$

For example, consider $5x^4$. Here a = 5 and $n = 4$. Then, according to the above formula

$$\frac{d}{dx}(5x^4) = (5)(4)x^{4 - 1} = 20x^3$$

Here, it is important to know the following two basic rules of differentiation that have been used to obtain the formula for $\frac{d}{dx}(ax^n)$.

1. Firstly, we have used *the power rule*, which is one of the most frequently used differentiation rules. It states that the derivative of x^n is n multiplied by x raised to power $n - 1$:

$$\frac{d}{dx}(x^n) = nx^{n - 1}$$

The power rule is applicable for any power: zero, positive, negative or a fraction.

2. Secondly, we have applied *the constant multiple rule* which states that the derivative of constant times a function $f(x)$ is constant times the derivative of the function:

$$\frac{d}{dx}[af(x)] = a\frac{d}{dx}[f(x)]$$

In case of the term ax^n, we have $f(x) = x^n$, and the constant multiple rule tells us that

$$\frac{d}{dx}(ax^n) = a\frac{d}{dx}(x^n)$$

Combining the constant multiple rule with the power rule, we get the desired formula for $\frac{d}{dx}(ax^n)$

$$\frac{d}{dx}(ax^n) = a\frac{d}{dx}(x^n) = a\left(nx^{n - 1}\right) = anx^{n - 1}$$

Recall that when the power n is either zero or a positive integer, ax^n is a typical term of a polynomial. Since the above formula for derivative of ax^n is valid for n being zero, positive number, negative number or a fraction, it is applicable not only to the case of polynomials but

also to more general terms having power of x being negative or a fraction.
The third basic rule of differentiation-known as *the addition rule* states that the derivative of the sum of a number of functions equals the sum of derivatives of these functions. Thus, if f_1, f_2, f_3, …, f_n are functions of x, then

$$\frac{d}{dx}(f_1 + f_2 + f_3 + \ldots + f_n) = \frac{df_1}{dx} + \frac{df_2}{dx} + \frac{df_3}{dx} + \ldots + \frac{df_n}{dx}$$

As an example, suppose we have a polynomial $f(x) = 5x^4 - 3x^3 + 2x^2 - 4$ having four terms. Let $f_1 = 5x^4$, $f_2 = -3x^3$, $f_3 = 2x^2$ and $f_4 = -4$, then noting that $5x^4 + (-3x^3) + 2x^2 + (-4) = 5x^4 - 3x^3 + 2x^2 - 4$, application of the addition rule gives

$$\frac{d}{dx}(5x^4 - 3x^3 + 2x^2 - 4) = \frac{d}{dx}(5x^4) + \frac{d}{dx}(-3x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(-4)$$

The fourth basic rule of differentiation-known as *the constant rule* states that the derivative of a constant c is zero,

$$\frac{d}{dx}c = 0$$

The four basic differentiation rules discussed above are applicable to all types of functions. In subsequent chapters, we will be applying them freely to other functions.
Before proceeding further, let us list below some important properties of exponents of real numbers.

$$x^m x^n = x^{m+n}, \qquad \frac{1}{x^n} = x^{-n}, \qquad \frac{x^m}{x^n} = x^m x^{-n} = x^{m-n}$$

$$(x^m)^n = x^{mn}, \qquad (cx)^n = c^n x^n, \qquad \sqrt{cx} = (cx)^{1/2} = c^{1/2}x^{1/2} = \sqrt{c}\sqrt{x}$$

$$x^{m/n} = (x^{1/n})^m = (\sqrt[n]{x})^m = (x^m)^{1/n} = \sqrt[n]{x^m}$$

It is useful to recall the following:

- When no multiplying coefficient or exponent is explicitly written, the same equals 1. Thus, for example, $x^2 = 1x^2$ and $x = 1x^1 = x^1$.
- $x^0 = 1$, $\frac{1}{x} = \frac{1}{x^1} = x^{-1}$, $\sqrt{x} = x^{1/2}$, $\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$

Example 1: Find the following derivative with respect to x.

$$\frac{d}{dx}(7x^4)$$

Solution: Comparing $7x^4$ with the general term ax^n, we find that $a = 7$ and $n = 4$. Applying the formula $\frac{d}{dx}(ax^n) = anx^{n-1}$, we have

$$\frac{d}{dx}(7x^4) = (7)(4)x^{4-1} = 28x^3$$

Example 2: Find the following derivative with respect to x.

$$\frac{d}{dx}(\sqrt{5x})$$

Solution: First, use the result $\sqrt{cx} = \sqrt{c}\,\sqrt{x}$, to write $\sqrt{5x} = \sqrt{5}\sqrt{x}$. Note that $\sqrt{x} = x^{1/2}$, so that $\frac{d}{dx}(\sqrt{5x}) = \frac{d}{dx}(\sqrt{5}\sqrt{x}) = \frac{d}{dx}(\sqrt{5}x^{1/2})$. Comparing $\sqrt{5}x^{1/2}$ with ax^n, we find that $a = \sqrt{5}$ and $n = \frac{1}{2}$. Applying the formula $\frac{d}{dx}(ax^n) = anx^{n-1}$, we have

$$\frac{d}{dx}(\sqrt{5x}) = \frac{d}{dx}(\sqrt{5}x^{1/2}) = (\sqrt{5})\left(\frac{1}{2}\right)x^{1/2-1} = \frac{\sqrt{5}}{2}x^{-1/2}$$

Since fractions can be subtracted only when they have common denominator, we have written $\frac{1}{2} - 1 = \frac{1}{2} - \frac{1}{1} = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$. Also, since $x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$, we can write the answer as

$$\frac{d}{dx}(\sqrt{5x}) = \frac{\sqrt{5}}{2}x^{-1/2} = \frac{\sqrt{5}}{2\sqrt{x}}$$

The above answer is correct. However, if we wish, we can rationalize the denominator by multiplying with $\frac{\sqrt{x}}{\sqrt{x}}$ to rewrite the answer as follows

$$\frac{d}{dx}(\sqrt{5x}) = \frac{\sqrt{5}}{2\sqrt{x}} = \frac{\sqrt{5}\,\sqrt{x}}{2\sqrt{x}\sqrt{x}} = \frac{\sqrt{5x}}{2x}$$

Note that we have written $\sqrt{5}\sqrt{x} = \sqrt{5x}$ and $\sqrt{x}\sqrt{x} = \sqrt{x^2} = x$.

Example 3: Find the following derivative with respect to t.

$$\frac{d}{dt}\left(3t^{1/3}\right)$$

Solution: Here, the variable is *t*. We, therefore, need to replace *x* with *t* in various formulas.

Comparing $3t^{1/3}$ with at^n, we find that $a = 3$ and $n = \frac{1}{3}$. Applying the formula $\frac{d}{dt}\left(at^n\right) = ant^{n - 1}$ we have

$$\frac{d}{dt}\left(3t^{1/3}\right) = (3)\left(\frac{1}{3}\right)t^{1/3 - 1} = t^{-2/3}$$

Note that we have written $\frac{1}{3} - 1 = \frac{1}{3} - \frac{1}{1} = \frac{1}{3} - \frac{3}{3} = \frac{1 - 3}{3} = -\frac{2}{3}$ (subtract fractions with common denominator)

Since $t^{-2/3} = \frac{1}{t^{2/3}}$, we can rewrite the answer as

$$\frac{d}{dt}\left(3t^{1/3}\right) = t^{-2/3} = \frac{1}{t^{2/3}}$$

Example 4: Find the following derivative with respect to u.

$$\frac{d}{du}\left(\frac{6}{u}\right)$$

Solution: Here, the variable is *u*. We, therefore, need to replace *x* with *u* in various formulas.

Using the result $\frac{6}{u} = 6u^{-1}$ and comparing $6u^{-1}$ with au^n, we find that $a = 6$ and $n = -1$. Use of formula $\frac{d}{du}\left(au^n\right) = anu^{n - 1}$ gives

$$\frac{d}{du}\left(\frac{6}{u}\right) = \frac{d}{du}\left(6u^{-1}\right) = (6)(-1)u^{-1-1}$$

$$= -6u^{-2} = -\frac{6}{u^2}$$

Note that we have written $(6)(-1)=-6$, $-1-1=-2$ and $u^{-2}=\frac{1}{u^2}$.

Example 5: Find following derivative with respect to x.

$$\frac{d}{dx}(3x^3-4x)$$

Solution: In this case, we will use the addition rule, which says that for functions f_1 and f_2,

$$\frac{d}{dx}(f_1+f_2)=\frac{df_1}{dx}+\frac{df_2}{dx}$$

Substituting $f_1=3x^3,\ f_2=-4x$, we have

$$\frac{d}{dx}[3x^3+(-4x)]=\frac{d}{dx}(3x^3)+\frac{d}{dx}(-4x)$$

We now need to calculate two derivatives: $\frac{d}{dx}(3x^3)$ and $\frac{d}{dx}(-4x)$.

To find $\frac{d}{dx}(3x^3)$, compare $3x^3$ with ax^n to find that $a=3$ and $n=3$. Substitute these values in the formula $\frac{d}{dt}(ax^n)=anx^{n-1}$ to get

$$\frac{d}{dx}(3x^3)=(3)(3)x^{3-1}=9x^2$$

To find $\frac{d}{dx}(-4x)$, first recall that $x=x^1$ and then compare $-4x^1$ with ax^n to find that $a=-4$ and $n=1$. Substitute these values in the formula $\frac{d}{dt}(ax^n)=anx^{n-1}$ and noting that $x^0=1$, we get

$$\frac{d}{dx}(-4x)=(-4)(1)x^{1-1}=-4x^0=-4(1)=-4$$

Substituting the values of $\frac{d}{dx}(3x^3)$ and $\frac{d}{dx}(-4x)$, we have

$$\frac{d}{dx}[3x^3+(-4x)]=\frac{d}{dx}(3x^3)+\frac{d}{dx}(-4x)=9x^2+(-4)$$

We can rewrite the above result to get the final answer as

$$\frac{d}{dx}(3x^3 - 4x) = 9x^2 - 4$$

Example 6: Find following derivative with respect to x.

$$\frac{d}{dx}\left(8\sqrt{x} + \frac{3}{x}\right)$$

Solution: In this case, we will use the addition rule, which says that for functions f_1 and f_2,
$\frac{d}{dx}(f_1 + f_2) = \frac{df_1}{dx} + \frac{df_2}{dx}$.

Substituting $f_1 = 8\sqrt{x}$, $f_2 = \frac{3}{x}$, we have

$$\frac{d}{dx}\left(8\sqrt{x} + \frac{3}{x}\right) = \frac{d}{dx}(8\sqrt{x}) + \frac{d}{dx}\left(\frac{3}{x}\right)$$

We now need to calculate two derivatives: $\frac{d}{dx}(8\sqrt{x})$ and $\frac{d}{dx}\left(\frac{3}{x}\right)$.

Comparing $8\sqrt{x}$ with ax^n and recalling the result $\sqrt{x} = x^{1/2}$, we find that $a = 8$ and $n = 1/2$. Substitute these values in the formula $\frac{d}{dt}(ax^n) = anx^{n - 1}$ to get

$$\frac{d}{dx}(8\sqrt{x}) = \frac{d}{dx}(8x^{1/2}) = (8)\left(\frac{1}{2}\right)x^{1/2 - 1} = 4x^{-1/2}$$

Note that we have written $\frac{1}{2} - 1 = \frac{1}{2} - \frac{1}{1} = \frac{1}{2} - \frac{2}{2} = \frac{1 - 2}{2} = -\frac{1}{2}$, because fractions can be added or subtracted only when they have common denominator.

To find $\frac{d}{dx}\left(\frac{3}{x}\right)$, we first use the result $\frac{3}{x} = 3x^{-1}$ and then compare $3x^{-1}$ with ax^n to find that $a = 3$ and $n = -1$. Substitute these values in the formula $\frac{d}{dt}(ax^n) = anx^{n - 1}$ to get

$$\frac{d}{dx}\left(\frac{3}{x}\right) = \frac{d}{dx}(3x^{-1}) = (3)(-1)x^{-1-1} = -3x^{-2}$$

Using the values of $\frac{d}{dx}(8\sqrt{x})$ and $\frac{d}{dx}\left(\frac{3}{x}\right)$, we get the result

$$\frac{d}{dx}\left(8\sqrt{x} + \frac{3}{x}\right) = 4x^{-1/2} + (-3x^{-2}) = \frac{4}{x^{1/2}} - \frac{3}{x^2} = \frac{4}{\sqrt{x}} - \frac{3}{x^2}$$

The above answer is correct. However, if we wish, we can rationalize the denominator of $\frac{4}{\sqrt{x}}$ by multiplying with $\frac{\sqrt{x}}{\sqrt{x}}$ to rewrite the answer as follows

$$\frac{d}{dx}\left(8\sqrt{x} + \frac{3}{x}\right) = \frac{4\sqrt{x}}{\sqrt{x}\sqrt{x}} - \frac{3}{x^2} = \frac{4\sqrt{x}}{x} - \frac{3}{x^2}$$

Note that we have written $\sqrt{x}\sqrt{x} = \sqrt{x^2} = x$.

Chapter 1 Exercises

Set A

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1 $\frac{d}{dx}(7x^3) =$

2

$\frac{d}{dx}[(4x)^2] =$

3

$\frac{d}{dx}\left(\frac{5}{\sqrt{x}}\right) =$

4

$\frac{d}{dx}\left(\frac{4}{x^{3/7}}\right) =$

5

$\frac{d}{dt}(5\sqrt{t}) =$

6

$\frac{d}{dt}\left(\frac{100}{t^{100}}\right) =$

$\frac{d}{dx}(8x^{2/3}) =$

$\frac{d}{dx}\left(\frac{x^{4/7}}{6}\right)$

Set B

$$\frac{d}{dx}\left(5x^4 - 3x^3 + 2x - 4\right) =$$

10

$$\frac{d}{dt}\left(7t^2 + t + 1\right) =$$

11

$$\frac{d}{dx}\left(10x^{10} + \frac{10}{\pi^{10}}\right)$$

12

$$\frac{d}{dx}\left(4x^4 + 3x^3 + \frac{4}{x}\right) =$$

13

$$\frac{d}{dx}\left(2 + x\right) =$$

14

$$\frac{d}{du}\left(6x^{7/6} + 7x^{5/6} + 9\right) =$$

$$\frac{d}{du}\left(8\sqrt{u} + u\right)$$

Chapter 1 Solutions

Set A

1

$$\frac{d}{dx}(7x^3) = (7)(3)x^{3-1} = 21x^2$$

2

$$\frac{d}{dx}[(4x)^2] = \frac{d}{dx}(4^2x^2) = \frac{d}{dx}(16x^2) = (16)(2)x^{2-1} = 32x$$

Explanation:

- Since $(cx)^n = c^nx^n$, we have $(4x)^2 = 4^2x^2$.

3

$$\frac{d}{dx}\left(\frac{5}{\sqrt{x}}\right) = \frac{d}{dx}\left[\frac{5}{x^{1/2}}\right] = \frac{d}{dx}(5x^{-1/2}) = (5)\left(-\frac{1}{2}\right)x^{-1/2-1} = -\frac{5}{2}x^{-3/2}$$

$$= -\frac{5}{2x^{3/2}} = -\frac{5}{2xx^{1/2}} = -\frac{5}{2x\sqrt{x}} = -\frac{5}{2x\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = = -\frac{5\sqrt{x}}{2xx} = -\frac{5\sqrt{x}}{2x^2}$$

Explanation:

- Since $\frac{1}{x^n} = x^{-n}$, we have $\frac{1}{x^{1/2}} = x^{-1/2}$.
- Recalling that fractions can be added or subtracted only when they have common denominator, we have $-\frac{1}{2} - 1 = -\frac{1}{2} - \frac{2}{2} = \frac{-1-2}{2} = -\frac{3}{2}$.
- Since $x^{m+n} = x^mx^n$, we have $x^{3/2} = x^{1+1/2} = x^1x^{1/2} = x\sqrt{x}$.
- We have rationalized the denominator by multiplying $-\frac{5}{2x\sqrt{x}}$ by $\frac{\sqrt{x}}{\sqrt{x}}$.
- $\sqrt{x}\sqrt{x} = \sqrt{x^2} = x.$
- $-\frac{5}{2}x^{-3/2}$, $-\frac{5}{2x^{3/2}}$, $-\frac{5}{2x\sqrt{x}}$ and $-\frac{5\sqrt{x}}{2x^2}$ are all correct answers. However, only $-\frac{5\sqrt{x}}{2x^2}$ has rational denominator.

4

$$\frac{d}{dx}\left(\frac{4}{x^{3/7}}\right) = \frac{d}{dx}\left(4x^{-3/7}\right) = (4)\left(-\frac{3}{7}\right)x^{-3/7-1} = -\frac{12}{7}x^{-10/7} = -\frac{12}{7x^{10/7}}$$

Explanation:

- Since $\frac{1}{x^n} = x^{-n}$, we have $\frac{1}{x^{3/7}} = x^{-3/7}$.
- $-\frac{3}{7} - 1 = -\frac{3}{7} - \frac{7}{7} = \frac{-3-7}{7} = -\frac{10}{7}$ (fractions can be added or subtracted only when they have common denominator).
- Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-10/7} = \frac{1}{x^{10/7}}$.
- Both $-\frac{12}{7}x^{-10/7}$ and $-\frac{12}{7x^{10/7}}$ are correct answers.

5

$$\frac{d}{dt}(5\sqrt{t}) = \frac{d}{dt}\left(5t^{1/2}\right) = (5)\left(\frac{1}{2}\right)t^{1/2-1} = \frac{5}{2}t^{-\frac{1}{2}} = \frac{5}{2t^{1/2}} = \frac{5}{2\sqrt{t}} = \frac{5}{2\sqrt{t}}\frac{\sqrt{t}}{\sqrt{t}} = \frac{5\sqrt{t}}{2t}$$

Explanation:

- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (fractions can be added or subtracted only when they have common denominator).
- Since $t^{-n} = \frac{1}{t^n}$, we have $t^{-1/2} = \frac{1}{t^{1/2}}$.
- $\frac{5}{2}t^{-1/2}$, $\frac{5}{2\sqrt{t}}$ and $\frac{5\sqrt{t}}{2t}$ are all correct answers. However, only $\frac{5\sqrt{t}}{2t}$ has rational denominator.
- $\sqrt{t}\sqrt{t} = \sqrt{t^2} = t$.

6

$$\frac{d}{dt}\left(\frac{100}{t^{100}}\right) = \frac{d}{dt}\left(100t^{-100}\right) = (100)(-100)t^{-100-1} = -10000t^{-101} = -\frac{10000}{t^{101}}$$

Explanation:

- Since $t^{-n} = \frac{1}{t}$, we have $t^{-100} = \frac{1}{t^{100}}$.
- $-100 - 1 = -101$.
- Both $-10000t^{-101}$ and $-\frac{10000}{t^{101}}$ are correct answers.

7

$$\frac{d}{dx}(8x^{2/3}) = (8)\left(\frac{2}{3}\right)x^{2/3 - 1} = \frac{16}{3}x^{-1/3} = \frac{16}{3x^{1/3}}$$

Explanation:

- $\frac{2}{3} - 1 = \frac{2}{3} - \frac{3}{3} = \frac{2 - 3}{3} = -\frac{1}{3}$ (fractions can be subtracted only when they have common denominator).
- Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-1/3} = \frac{1}{x^{1/3}}$.
- Both $\frac{16}{3}x^{-1/3}$ and $\frac{16}{3x^{1/3}}$ are correct answers.

8

$$\frac{d}{dx}\left(\frac{x^{4/7}}{6}\right) = \left(\frac{1}{6}\right)\left(\frac{4}{7}\right)x^{4/7 - 1} = \frac{4}{42}x^{-3/7} = \frac{2}{21}x^{-3/7} = \frac{2}{21x^{3/7}}$$

Explanation:

- $\frac{4}{7} - 1 = \frac{4}{7} - \frac{7}{7} = \frac{4 - 7}{7} = -\frac{3}{7}$ (fractions can be subtracted only when they have common denominator).
- Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-3/7} = \frac{1}{x^{3/7}}$.
- $\frac{4}{42} = \frac{2 \times 2}{2 \times 21} = \frac{2}{21}$.
- Both $\frac{2}{21}x^{-3/7}$ and $\frac{2}{21x^{3/7}}$ are correct answers.

Set B

9

Using addition rule, we have

$$\frac{d}{dx}(5x^4 - 3x^3 + 2x - 4) = \frac{d}{dx}(5x^4) + \frac{d}{dx}(-3x^3) + \frac{d}{dx}(2x^1) + \frac{d}{dx}(-4)$$

$$= (5)(4)x^{4-1} + (-3)(3)x^{3-1} + (2)(1)x^{1-1} + 0 = 20x^3 - 9x^2 + 2x^0$$

$$= 20x^3 - 9x^2 + 2(1) = 20x^3 - 9x^2 + 2$$

Explanation:

- Recall that $x = x^1$ and $x^0 = 1$.

- Since the derivative of a constant is zero, we have $\frac{d}{dx}(-4) = 0$.

10

$$\frac{d}{dt}(7t^2 + t + 1)$$
$$= \frac{d}{dt}(7t^2) + \frac{d}{dt}(t) + \frac{d}{dt}(1) = (7)(2)t^{2-1} + (1)(1)t^{1-1} + 0 = 14t + t^0$$
$$= 14t + 1$$

Explanation:

- Since the derivative of a constant is zero, we have $\frac{d}{dx}(1) = 0$.
- Note that $t = t^1 = 1t^1$ and $t^0 = 1$.

11

$$\frac{d}{dx}\left(10x^{10} + \frac{10}{\pi^{10}}\right) = \frac{d}{dx}(10x^{10}) + \frac{d}{dx}\left(\frac{10}{\pi^{10}}\right) = (10)(10)x^{10-1} + 0 = 100x^9$$

Explanation:

- Since the derivative of a constant is zero and $\left(\frac{10}{\pi^{10}}\right)$ is a constant, we have $\frac{d}{dx}\left(\frac{10}{\pi^{10}}\right) = 0$.

12

$$\frac{d}{dx}\left(4x^4 + 3x^3 + \frac{4}{x}\right) = \frac{d}{dx}(4x^4) + \frac{d}{dx}(3x^3) + \frac{d}{dx}(4x^{-1})$$

$$= (4)(4)x^{4-1} + (3)(3)x^{3-1} + (4)(-1)x^{-1-1} = 16x^3 + 9x^2 - 4x^{-2} = 16x^3 + 9x^2 - \frac{4}{x^2}$$

Explanation:

- Since $\frac{1}{x^n} = x^{-n}$, we have $\frac{1}{x} = x^{-1}$.
- $4(-1) =- 4, \quad -1 - 1 =- 2.$
- Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-2} = \frac{1}{x^2}$.
- Both $16x^3 + 9x^2 - 4x^{-2}$ and $16x^3 + 9x^2 - \frac{4}{x^2}$ are correct answers.

$$\frac{d}{dx}(2 + x) = \frac{d}{dx}(2) + \frac{d}{dx}(1x^1) = 0 + (1)(1)x^{1 - 1} = 1x^0 = 1$$

Explanation:

- Recall that $x^0 = 1$.

14

$$\frac{d}{dx}(6x^{7/6} + 7x^{5/6} + 9) = \frac{d}{dx}(6x^{7/6}) + \frac{d}{dx}(7x^{5/6}) + \frac{d}{dx}(9)$$

$$= (6)\left(\frac{7}{6}\right)x^{7/6 - 1} + (7)\left(\frac{5}{6}\right)x^{5/6 - 1} + 0 = 7x^{1/6} + \frac{35}{6}x^{-1/6} = 7x^{1/6} + \frac{35}{6x^{1/6}}$$

Explanation:

- Since the derivative of a constant is zero, we have $\frac{d}{dx}(9) = 0$.
- Recalling that fractions can be added or subtracted only when they have common denominator, we have $\frac{7}{6} - 1 = \frac{7}{6} - \frac{6}{6} = \frac{7 - 6}{6} = \frac{1}{6}$ and $\frac{5}{6} - 1 = \frac{5}{6} - \frac{6}{6} = \frac{5 - 6}{6} = -\frac{1}{6}$.
- Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-1/6} = \frac{1}{x^{-1/6}}$.
- Both $7x^{1/6} + \frac{35}{6}x^{-1/6}$ and $7x^{1/6} + \frac{35}{6x^{1/6}}$ are correct answers.

[redacted]

$$\frac{d}{du}(8\sqrt{u} + u) = \frac{d}{du}(8\sqrt{u}) + \frac{d}{du}(u) = \frac{d}{du}(8u^{1/2}) + \frac{d}{du}(1u^1)$$

$$= (8)\left(\frac{1}{2}\right)u^{1/2 - 1} + (1)(1)u^{1 - 1} = \frac{8}{2}u^{-1/2} + 1u^0$$

$$= 4\frac{1}{u^{1/2}} + u^0 = \frac{4}{\sqrt{u}} + 1 = \frac{4\sqrt{u}}{\sqrt{u}\sqrt{u}} + 1 = \frac{4\sqrt{u}}{u} + 1$$

Explanation:

- Note that $u = u^1 = 1u^1$ and $u^0 = 1$.
- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1 - 2}{2} = -\frac{1}{2}$ (fractions can be subtracted only when they have common denominator).
- Since $u^{-n} = \frac{1}{u^n}$, we have $u^{-1/2} = \frac{1}{u^{1/2}}$.

- Both $\frac{4}{\sqrt{u}} + 1$ and $\frac{4\sqrt{u}}{u} + 1$ are correct answers. However, only $\frac{4\sqrt{u}}{u} + 1$ has rational denominator.

2 THE CHAIN RULE

Perhaps the most important among the basic rules of differentiation is known as the chain rule. To understand the use of the chain rule, let us consider an example. Suppose we wish to find the derivative of $f(x) = (5x^3 + 3)^4$. One way is to multiply $(5x^3 + 3)$ by itself four times and then find the derivative of the resulting polynomial function. Obviously, such a process is tedious and time consuming. As seen in Example 1 given below, the chain rule helps us to avoid this difficulty.

Let $y = f(u)$ be a function of the variable u. If $u = g(x)$ in turn is a function of the variable x, then, y itself becomes a function of another function u. Such a function is known as composite function and can be written as $y = f(g(x))$. In such a situation, we can find the derivative of y with respect to x by using the chain rule which can be stated as follows.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or

Derivative of y with respect to x = Derivative of y with respect to u × Derivative of u with respect to x

The chain rule can be stated in another way as follows. The function $y = f(u)$ is known as the outside function and $u(x)$ is known as inside function. Then

Derivative of y with respect to x = (derivative of the outside function with respect to inside function considered as a variable) × (derivative of inside function with respect to x)

As we will see in the present and subsequent chapters, the chain rule is one of the most widely used methods to find the derivatives. A few examples of the evaluating the derivatives where chain rule is applied are given below.

$$\frac{d}{dx}\frac{1}{\sqrt{8x^3 + 4x^2 + 7}}, \qquad \frac{d}{d\theta}\sqrt[3]{\sin\theta}, \qquad \frac{d}{dx}4e^{(5x^2 + 6)},$$

$$\frac{d}{dx}\ln|\sin x|$$

Study carefully the following Examples to learn how to use the chain rule and then attempt the Exercises given at the end of the Chapter.

Example 1: Find the following derivative with respect to x.

$$\frac{d}{dx}(5x^3 + 3)^4$$

Solution: We will apply the chain rule to find $\frac{d}{dx}(5x^3 + 3)^4$. Let $y = (5x^3 + 3)^4$

Put $u = 5x^3 + 3$, so that

$$y(u) = u^4, \qquad u(x) = 5x^3 + 3, \qquad \frac{d}{dx}(5x^3 + 3)^4 = \frac{dy}{dx} = \ ?$$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^4)\right]\left[\frac{d}{dx}(5x^3 + 3)\right] = (4u^3)\,(15x^2 + 0) = (4u^3)\,(15x^2)$$

Substitute $u = 5x^3 + 3$ in the above expression to obtain

$$\frac{dy}{dx} = 4(5x^3 + 3)^3(15x^2) = 60x^2(5x^3 + 3)^3$$

Example 2: Find the following derivative with respect to t.

$$\frac{d}{dt}\sqrt{7t^4 + 12}$$

Solution: Let $y = \sqrt{7t^4 + 12}$. Now, put $u = 7t^4 + 12$, so that

$$y(u) = \sqrt{u} = u^{1/2}, \quad u(t) = 7t^4 + 12, \quad \frac{d}{dt}\sqrt{7t^4 + 12} = \frac{dy}{dt} = \ ?$$

Applying the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = \left[\frac{d}{du}(u^{1/2})\right]\left[\frac{d}{dt}(7t^4 + 12)\right] = \left(\frac{1}{2}u^{-1/2}\right)(28t^3 + 0)$$

$$= \left(\frac{1}{2u^{1/2}}\right)(28t^3) = \frac{14t^3}{u^{1/2}} = \frac{14t^3}{\sqrt{u}}$$

Substitute $u = 7t^4 + 12$ in the above expression to obtain

$$\frac{dy}{dt} = \frac{14t^3}{\sqrt{7t^4 + 12}}$$

Note that we have written $\sqrt{u} = u^{1/2}$ and $u^{-1/2} = \frac{1}{u^{1/2}}$. The above answer is correct. However,

we can rationalize the denominator by multiplying with $\frac{\sqrt{7t^4 + 12}}{\sqrt{7t^4 + 12}}$ and using the result $\sqrt{7t^4 + 12}\sqrt{7t^4 + 12} = \sqrt{(7t^4 + 12)^2} = (7t^4 + 12)$ as follows:

$$\frac{dy}{dt} = \frac{14t^3 \quad \sqrt{7t^4 + 12}}{\sqrt{7t^4 + 12}\sqrt{7t^4 + 12}} = \frac{14t^3\sqrt{7t^4 + 12}}{(7t^4 + 12)}$$

Chapter 2 Exercises

Set A

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d}{dx}(\sqrt{x} + 1)^4 =$$

$$\frac{d}{dx}(3x^3 + 8x^2 + 3)^6 =$$

3

$$\frac{d}{dx}(2x^{7/2} + 6x^{3/2})^4 =$$

Set B

4

$$\frac{d}{dx}(4x^4 + 10\pi^6)^{1/4} =$$

5

$$\frac{d}{dx}\sqrt{3 + \sqrt{x}} =$$

6

$$\frac{d}{dx}\sqrt{3x^3 + 4} =$$

Chapter 2 Solutions

Set A

1 We will apply the chain rule to find $\frac{d}{dx}(\sqrt{x}+1)^4$. Let $y = (\sqrt{x}+1)^4$

Put $u = \sqrt{x}+1$, so that $y(u) = u^4,\ u(x) = \sqrt{x}+1, \qquad \frac{d}{dx}(\sqrt{x}+1)^4 = \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^4)\right]\left[\frac{d}{dx}(x^{1/2}+1)\right] = (4u^3)\left(\frac{1}{2}x^{-1/2}+0\right)$$

$$= (4u^3)\left(\frac{1}{2}x^{-1/2}\right) = (4u^3)\left(\frac{1}{2x^{1/2}}\right) = \frac{2u^3}{x^{1/2}} = \frac{2u^3}{\sqrt{x}} = \frac{2(\sqrt{x}+1)^3}{\sqrt{x}}$$

$$= \frac{2(\sqrt{x}+1)^3\sqrt{x}}{\sqrt{x}\quad\sqrt{x}} = \frac{2(\sqrt{x}+1)^3\sqrt{x}}{\sqrt{x^2}} = \frac{2\sqrt{x}(\sqrt{x}+1)^3}{x}$$

Explanation:

- $\sqrt{x} = x^{1/2}$. Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-1/2} = \frac{1}{x^{1/2}}$.
- Since the derivative of a constant is zero, we have $\frac{d}{dx}(1) = 0$.
- $\sqrt{x}\sqrt{x} = \sqrt{x^2} = x$.
- Both $\frac{2(\sqrt{x}+1)^3}{\sqrt{x}}$ and $\frac{2\sqrt{x}(\sqrt{x}+1)^3}{x}$ are correct answers. However, $\frac{2\sqrt{x}(\sqrt{x}+1)^3}{x}$ has rational denominator. We have rationalized the denominator by multiplying $\frac{2(\sqrt{x}+1)^3}{\sqrt{x}}$ by $\frac{\sqrt{x}}{\sqrt{x}}$.

Apply the chain rule to find $\frac{d}{dx}(3x^3+8x^2+3)^6$. Let $y = (3x^3+8x^2+3)^6$.

Put $u = 3x^3+8x^2+3$, so that $y(u) = u^6, u(x) = 3x^3+8x^2+3,\ \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^6)\right]\left[\frac{d}{dx}(3x^3+8x^2+3)\right] = (6u^5)(9x^2+16x+0)$$

$$= 6(3x^3+8x^2+3)^5(9x^2+16x) = 6(9x^2+16x)(3x^3+8x^2+3)^5 = 6x(9x+16)(3x^3+8x^2+3)^5$$

Explanation:

- $\frac{d}{dx}(3) = 0$ (Because the derivative of a constant is zero).

3

$$\frac{d}{dx}(2x^{7/2} + 6x^{3/2})^4 = ?$$

We will apply the chain rule to find $\frac{d}{dx}(2x^{7/2} + 6x^{3/2})^4$.

Let $y = \frac{d}{dx}(2x^{7/2} + 6x^{3/2})^4$. Put $u = 2x^{7/2} + 6x^{3/2}$, so that

$$y(u) = u^4, \quad u(x) = 2x^{7/2} + 6x^{3/2}, \qquad \frac{d}{dx}(2x^{7/2} + 6x^{3/2})^4 = \frac{dy}{dx} = ?$$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^4)\right]\left[\frac{d}{dx}(2x^{7/2} + 6x^{3/2})\right] = [4u^3]\left[\frac{d}{dx}(2x^{7/2}) + \frac{d}{dx}(6x^{3/2})\right]$$

$$= (4u^3)\left[(2)\left(\frac{7}{2}\right)x^{5/2} + (6)\left(\frac{3}{2}\right)x^{1/2}\right] = 4(u^3)(7x^{5/2} + 9x^{1/2}) = 4(2x^{7/2} + 6x^{3/2})^3(7x^{5/2} + 9x^{1/2})$$

Set B

4

$$\frac{d}{dx}(4x^4 + 10\pi^6)^{1/4} = ?$$

We will apply the chain rule to find $\frac{d}{dx}(4x^4 + 10\pi^6)^{1/4}$. Let $y = (4x^4 + 10\pi^6)^{1/4}$

Now, put $u = 4x^4 + 10\pi^6$, so that

$$y(u) = u^{1/4}, \quad u(x) = 4x^4 + 10\pi^6, \qquad \frac{d}{dx}(4x^4 + 10\pi^6)^{1/4} = \frac{dy}{dx} = ?$$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^{1/4})\right]\left[\frac{d}{dx}(4x^4 + 10\pi^6)\right]$$

$$= \left[\frac{d}{du}(u^{1/4})\right]\left[\frac{d}{dx}(4x^4) + \frac{d}{dx}(10\pi^6)\right]$$

$$= \left(\frac{1}{4}u^{-3/4}\right)(16x^3 + 0) = \left(\frac{1}{4u^{3/4}}\right)(16x^3) = \frac{16x^3}{4u^{3/4}} = \frac{4x^3}{u^{3/4}} = \frac{4x^3}{(4x^4 + 10\pi^6)^{3/4}}$$

Explanation:

- Since $x^{-n} = \frac{1}{x^n}$, we have $u^{-3/4} = \frac{1}{4u^{3/4}}$.
- Since the derivative of a constant is zero and $(10\pi^6)$ is a constant, we have $\frac{d}{dx}(10\pi^6) = 0$.

We will apply the chain rule to find $\frac{d}{dx}\sqrt{3+\sqrt{x}}$. Let $y = \sqrt{3+\sqrt{x}}$

Put $u = 3 + \sqrt{x}$, so that

$y(u) = \sqrt{u} = u^{1/2}$, $u(x) = 3 + \sqrt{x}$, $\frac{d}{dx}\sqrt{3+\sqrt{x}} = \frac{dy}{dx} =$?

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^{1/2})\right]\left[\frac{d}{dx}(3+\sqrt{x})\right] = \left(\frac{1}{2}u^{-1/2}\right)\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{1}{4u^{1/2}x^{1/2}} = \frac{1}{4x^{1/2}u^{1/2}} = \frac{1}{4\sqrt{x}\left(\sqrt{3+\sqrt{x}}\right)} = \frac{1}{4\left(\sqrt{x(3+\sqrt{x})}\right)} = \frac{1}{4\sqrt{3x+x\sqrt{x}}}$$

$$= \frac{1}{4\sqrt{3x+x\sqrt{x}}}\frac{\sqrt{3x+x\sqrt{x}}}{\sqrt{3x+x\sqrt{x}}} = \frac{\sqrt{3x+x\sqrt{x}}}{4(3x+x\sqrt{x})} = \frac{\sqrt{3x+x\sqrt{x}}\,(3x-x\sqrt{x})}{4(3x+x\sqrt{x})(3x-x\sqrt{x})}$$

$$= \frac{\sqrt{3x+x\sqrt{x}}\,(3x-x\sqrt{x})}{4\left[(3x)^2-(x\sqrt{x})^2\right]} = \frac{\sqrt{3x+x\sqrt{x}}\,(3x-x\sqrt{x})}{4\left[(3x)^2-x^2\sqrt{x}\sqrt{x}\right]}$$

$$= \frac{\left(\sqrt{3x+x\sqrt{x}}\right)(3x-x\sqrt{x})}{4(9x^2-x^3)}$$

Explanation:

- $\frac{1}{4\sqrt{x(3+\sqrt{x})}}$, $\frac{\sqrt{3x+x\sqrt{x}}}{4(3x+x\sqrt{x})}$ and $\frac{\left(\sqrt{3x+x\sqrt{x}}\right)(3x-x\sqrt{x})}{4(9x^2-x^3)}$ are all correct answers. However, only $\frac{\left(\sqrt{3x+x\sqrt{x}}\right)(3x-x\sqrt{x})}{4(9x^2-x^3)}$ has rational denominator.

6 We will apply the chain rule to find $\frac{d}{dx}\sqrt{3x^3+4}$. Let $y = \sqrt{3x^3+4}$

Put $u = 3x^3 + 4$, so that

$y(u) = \sqrt{u} = u^{1/2}$, $u(x) = 3x^3 + 4$, $\frac{d}{dx}\sqrt{3x^3+4} = \frac{dy}{dx} =$?

Apply the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^{1/2})\right]\left[\frac{d}{dx}(3x^3+4)\right]$

$$= \left(\frac{1}{2u^{1/2}}\right)(9x^2) = \frac{9x^2}{2\sqrt{u}} = \frac{9x^2}{2\sqrt{3x^3+4}} = \frac{9x^2}{2\sqrt{3x^3+4}}\frac{\sqrt{3x^3+4}}{\sqrt{3x^3+4}} = \frac{9x^2\sqrt{3x^3+4}}{2(3x^3+4)}$$

Explanation:

- $\sqrt{u} = u^{1/2}$. Since $x^{-n} = \frac{1}{x^n}$, we have $u^{-1/2} = \frac{1}{u^{1/2}}\,\frac{d}{dx}(4) = 0$ (because the derivative of a constant is zero).
- Both $\frac{9x^2}{2\sqrt{3x^3+4}}$ and $\frac{9x^2\sqrt{3x^3+4}}{2(3x^3+4)}$ are correct answers. However, only $\frac{9x^2\sqrt{3x^3+4}}{2(3x^3+4)}$ has

rational denominator.

3 THE PRODUCT RULE

The next basic rule of differentiation that is applicable to all types of functions is known as the Product Rule. It is used to find the derivatives of products of two functions $f(x)$ and $g(x)$ of the same variable x: $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$

Dropping the dependence on x, the product rule may be stated as follows.

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

Derivative of products of two functions = First function × (Derivatives of second function) + Second function × (Derivatives of First function)

We list below a few examples from the present and the subsequent chapters where product rule is used.

$\frac{d}{dx}(4x^2 + 3)(\sqrt{x} + 2)$, $\frac{d}{dx}\sec x \csc x$, $\frac{d}{dx}\tan^{-1} x \cot^{-1} x$, $\frac{d}{dx}x^2 \ln x$

Example 1: Find the following derivative with respect to x.

$$\frac{d}{dx}(6x^3 + 5)(x^4 + 6)$$

Solution: We will apply the product rule to find $\frac{d}{dx}(6x^3 + 5)(x^4 + 6)$. For this, let

$f(x) = 6x^3 + 5$, $g(x) = x^4 + 6$, so that $\frac{d}{dx}(6x^3 + 5)(x^4 + 6) = \frac{d}{dx}(fg) =$?

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = (6x^3 + 5)\left[\frac{d}{dx}(x^4 + 6)\right] + (x^4 + 6)\left[\frac{d}{dx}(6x^3 + 5)\right]$$

$$= (6x^3 + 5)(4x^3 + 0) + (x^4 + 6)(18x^2 + 0) = (6x^3 + 5)(4x^3) + (x^4 + 6)(18x^2)$$

$$= 24x^6 + 20x^3 + 18x^6 + 108x^2 = 42x^6 + 20x^3 + 108x^2$$

Let us now find the derivative by multiplying $(6x^3 + 5)(x^4 + 6)$ and then finding the derivative:

$$\frac{d}{dx}(6x^3 + 5)(x^4 + 6) = \frac{d}{dx}(6x^7 + 5x^4 + 36x^3 + 30) = 42x^6 + 20x^3 + 108x^2$$

We find that we get the same result as obtained by use of the product rule.

Example 2: Find the following derivative with respect to x.

$$\frac{d}{dx}(4x^2 + 3)(\sqrt{x} + 2)$$

Solution: We will apply the product rule to find $\frac{d}{dx}(4x^2 + 3)(\sqrt{x} + 2)$.

For this, let

$$f(x) = 4x^2 + 3, \ g(x) = \sqrt{x} + 2, \ \text{so that} \ \frac{d}{dx}(4x^2 + 3)(\sqrt{x} + 2) = \frac{d}{dx}(fg) = \ ?$$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = (4x^2 + 3)\left[\frac{d}{dx}(x^{1/2} + 2)\right] + (x^{1/2} + 2)\left[\frac{d}{dx}(4x^2 + 3)\right]$$

$$= (4x^2 + 3)\left[\frac{1}{2}x^{-1/2} + 0\right] + (x^{1/2} + 2)[8x + 0]$$

$$= (4x^2 + 3)\left(\frac{1}{2}x^{-1/2}\right) + (x^{1/2} + 2)(8x)$$

Distributing the above expression by using the result $(a + b)c = ab + ac$, we have

$$\frac{d}{dx}(fg) = \frac{4}{2}x^2x^{-1/2} + \frac{3}{2}x^{-1/2} + 8x^{1/2}x + 16x = 2x^{3/2} + \frac{3}{2}x^{-1/2} + 8x^{3/2} + 16x$$

$$= 10x^{3/2} + \frac{3}{2x^{1/2}} + 16x = 10x^{3/2} + \frac{3}{2\sqrt{x}} + 16x$$

$$= 10x^{3/2} + \frac{3\ \sqrt{x}}{2\sqrt{x}\sqrt{x}} + 16x = 10x^{3/2} + \frac{3\sqrt{x}}{2x} + 16x$$

Note that we have written $\sqrt{x} = x^{1/2}$, $x^{-1/2} = \frac{1}{x^{1/2}}$.

Since $x^m x^n = x^{m+n}$ and since fractions can be subtracted only when they have common denominator, we have written $x^2x^{-1/2} = x^{2+(-1/2)} = x^{2-1/2} = x^{\frac{4}{2}-\frac{1}{2}} = x^{\frac{4-1}{2}} = x^{3/2}$ and $x^{1/2}x = x^{\frac{1}{2}+1} = x^{\frac{1}{2}+\frac{2}{2}} = x^{\frac{1+2}{2}} = x^{3/2}$.

Also, we have rationalized the denominator of $\frac{3}{2\sqrt{x}}$ by multiplying with $\frac{\sqrt{x}}{\sqrt{x}}$ and using the result $\sqrt{x}\sqrt{x} = \sqrt{x^2} = x$.

Example 3: Find the following derivative with respect to x.

$$\frac{d}{dx}(x^2 + 5)^5(2x^2 + 5)$$

Solution: We will apply the product rule to find $\frac{d}{dx}(x^2 + 5)^5(2x^2 + 5)$. For this, let

$$f(x) = (x^2 + 5)^5, \ g(x) = (2x^2 + 5), \ \text{so that} \ \frac{d}{dx}(x^2 + 5)^5(2x^2 + 5) = \frac{d}{dx}(fg) = \ ?$$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = (x^2 + 5)^5\left[\frac{d}{dx}(2x^2 + 5)\right] + (2x^2 + 5)\left[\frac{d}{dx}(x^2 + 5)^5\right]$$

$$= (x^2 + 5)^5(4x + 0) + (2x^2 + 5)\left[\frac{d}{dx}(x^2 + 5)^5\right] = (x^2 + 5)^5(4x) + (2x^2 + 5)\left[\frac{d}{dx}(x^2 + 5)^5\right]$$

For finding the derivative of $\frac{d}{dx}(x^2 + 5)^5\left[= \frac{df}{dx}\right]$, we need to use the chain rule as follows.

Put $u = x^2 + 5$, so that $f(u) = u^5, \quad u(x) = x^2 + 5, \quad \frac{df}{dx} = \ ?$

Now, applying the chain rule, we have

$$\frac{df}{dx} = \frac{df}{du}\,\frac{du}{dx} = \left[\frac{d}{du}(u^5)\right]\left[\frac{d}{dx}(x^2 + 5)\right] = (5u^4)\ (2x + 0)$$

$$= \left[5(x^2 + 5)^4\right](2x) = 10x(x^2 + 5)^4$$

Substituting the above value of $\frac{d}{dx}(x^2 + 5)^5\left[= \frac{df}{dx}\right]$ in the expression for $\frac{d}{dx}(fg)$, we have

$$\frac{d}{dx}(fg) = (x^2 + 5)^5(4x) + (2x^2 + 5)\left[10x(x^2 + 5)^4\right] = 4x(x^2 + 5)^5 + 10x(2x^2 + 5)(x^2 + 5)^4$$

The above answer is correct. However, if we wish, we can simplify it further by factoring out $2x(x^2 + 5)^4$.

$$\frac{d}{dx}(fg) = 2x(x^2 + 5)^4\left[2(x^2 + 5) + 5(2x^2 + 5)\right]$$

$$= 2x(x^2 + 5)^4\left[2x^2 + 10 + 10x^2 + 25\right] = 2x(x^2 + 5)^4(12x^2 + 35)$$

Hence,

$$\frac{d}{dx}(x^2 + 5)^5(2x^2 + 5) = 2x(x^2 + 5)^4(10x^4 + 77x^2 + 135)$$

Chapter 3 Exercises

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d}{dx}8\sqrt{x}(3x + 5) =$$

2

$$\frac{d}{dx}(10x^{10} + x)(2x^2 + 3) =$$

3

$$\frac{d}{dx}(8x^3 + 7)^3(5x - 12)^4 =$$

Chapter 3 Solutions

1 We will apply the product rule to find $\frac{d}{dx}8\sqrt{x}(3x+5)$. For this, let

$f(x) = 8\sqrt{x} = 8x^{1/2}, \;\; g(x) = 3x + 5, \;\; \text{so that } \frac{d}{dx}8x^{1/2}(3x+5) = \frac{d}{dx}(fg) = ?$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = 8x^{1/2}\left[\frac{d}{dx}(3x+5)\right] + (3x+5)\left[\frac{d}{dx}\left(8x^{1/2}\right)\right]$$

$$= 8x^{1/2}(3+0) + (3x+5)\left(4x^{-1/2}\right) = 24x^{1/2} + 3x\left(4x^{-1/2}\right) + 5\left(4x^{-1/2}\right)$$

$$= 24x^{1/2} + 12xx^{-1/2} + 20x^{-1/2} = 24x^{1/2} + 12x^{1/2} + 20x^{-1/2}$$

$$= 36x^{1/2} + 20x^{-1/2} = 36\sqrt{x} + \frac{20}{\sqrt{x}} = 36\sqrt{x} + \frac{20\sqrt{x}}{\sqrt{x}\sqrt{x}} = 36\sqrt{x} + \frac{20\sqrt{x}}{x}$$

Explanation:

- Note that $\sqrt{x} = x^{1/2}$.
- Since $x^m x^n = x^{m+n}$ and since fractions can be subtracted only when they have common denominator, $xx^{-\frac{1}{2}} = x^1 x^{-1/2} = x^{1+\left(-\frac{1}{2}\right)} = x^{1-\frac{1}{2}} = x^{\frac{2}{2}-\frac{1}{2}} = x^{\frac{2-1}{2}} = x^{\frac{1}{2}}$.
- $x^{-1/2} = \frac{1}{x^{1/2}}$ and $\sqrt{x}\sqrt{x} = \sqrt{x^2} = x$.
- Both $36\sqrt{x} + \frac{20}{\sqrt{x}}$ and $36\sqrt{x} + \frac{20\sqrt{x}}{x}$ are correct answers. However, $36\sqrt{x} + \frac{20\sqrt{x}}{x}$ has rational denominator.

[redacted] We will apply the product rule to find $\frac{d}{dx}\left(10x^{10} + x\right)\left(2x^2 + 3\right)$. For this, let

$f(x) = 10x^{10} + x, \; g(x) = 2x^2 + 3 \text{ so that } \frac{d}{dx}\left(10x^{10} + x\right)\left(2x^2 + 3\right) = \frac{d}{dx}(fg) = ?$

Applying the product rule, we have $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$= \left(10x^{10} + x\right)\left[\frac{d}{dx}\left(2x^2 + 3\right)\right] + \left(2x^2 + 3\right)\left[\frac{d}{dx}\left(10x^{10} + x\right)\right]$$

$$= \left(10x^{10} + x\right)(4x) + \left(2x^2 + 3\right)\left(100x^9 + 1\right) = 40x^{11} + 4x^2 + 2x^2\left(100x^9 + 1\right) + 3\left(100x^9 + 1\right)$$

$$= 40x^{11} + 4x^2 + 200x^{11} + 2x^2 + 300x^9 + 3 = 240x^{11} + 300x^9 + 6x^2 + 3$$

Explanation:

- Since $x^m x^n = x^{m+n}$ and $x = x^1$, we have
  $x^{10}x = x^{10+1} = x^{11}, \ xx = x^{1+1} = x^2$ and $x^2x^9 = x^{2+9} = x^{11}$.

3 First, apply the product rule. For this, let
$f(x) = (8x^3 + 7)^3, \ g(x) = (5x - 12)^4$

$$\frac{d}{dx}(8x^3 + 7)^3(5x - 12)^4 = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$= (8x^3 + 7)^3\left[\frac{d}{dx}(5x - 12)^4\right] + (5x - 12)^4\left[\frac{d}{dx}(8x^3 + 7)^3\right]$

Next, apply the chain rule to find $\frac{d}{dx}(5x - 12)^4\left(= \frac{dg}{dx}\right)$ and $\frac{d}{dx}(8x^3 + 7)^3\left(= \frac{df}{dx}\right)$. Put
$u = 8x^3 + 7$, $v = 5x - 12$ so that $f(u) = u^3, \ \ u(x) = 8x^3 + 7$

$g(v) = v^4, v(x) = 5x - 12, \ \frac{d}{dx}(5x - 12)^4 = \frac{dg}{dx} = \ ?$ and $\frac{d}{dx}(8x^3 + 7)^3 = \frac{df}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{d}{dx}(fg) = (8x^3 + 7)^3\left[\frac{dg}{dv}\frac{dv}{dx}\right] + (5x - 12)^4\left[\frac{df}{du}\frac{du}{dx}\right]$$

$= (8x^3 + 7)^3\left[\frac{d}{du}(v^4)\frac{d}{dx}(5x - 12)\right] + (5x - 12)^4\left[\frac{d}{du}(u^3)\frac{d}{dx}(8x^3 + 7)\right]$

$= (8x^3 + 7)^3[(4v^3)\,(5)] + (5x - 12)^4[(3u^2)\,(24x^2)]$

$= (8x^3 + 7)^3[20v^3] + (5x - 12)^4[72x^2u^2]$

$= (8x^3 + 7)^3[20(5x - 12)^3] + (5x - 12)^4[72x^2(8x^3 + 7)^2]$

$= 20\mathbf{(}8x^3 + 7\mathbf{)}^3\mathbf{(}5x - 12\mathbf{)}^3 + 72x^2\mathbf{(}8x^3 + 7\mathbf{)}^2\mathbf{(}5x - 12\mathbf{)}^4$

The above answer is correct. However, if we wish, we can simplify it further by factoring out $(8x^3 + 7)^2(5x - 12)^3$ as follows.

$\frac{d}{dx}(fg) = (8x^3 + 7)^2(5x - 12)^3[20\,(8x^3 + 7) + \ 72x^2(5x - 12)]$

$= (8x^3 + 7)^2(5x - 12)^3[160x^3 \ + 140 + \ 360x^3 - 864x^2\ ]$

$= (8x^3 + 7)^2(5x - 12)^3 4[40x^3 \ + 35 + \ 90x^3 - 216x^2\ ]$

$$= 4(5x - 12)^3(8x^3 + 7)^2(130x^3 - 216x^2 + 35)$$

Explanation: Since the derivative of a constant is zero, we have $\frac{d}{dx}(7) = 0$ and $\frac{d}{dx}(12) = 0$

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4 THE QUOTIENT RULE

In this chapter, we will study the Quotient Rule, which is the last among the basic rules of differentiation that we are studying to find the derivatives of different types of differentiable functions. It is used to find the derivatives of a function, which is a quotient of two functions $f(x)$ and $g(x)$ of the same variable x:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{[g(x)]^2}$$

If we drop the dependence on x, the quotient rule may be stated as follows

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

There is an easy way to remember the quotient rule as follows. Let us denote the function $f(x)$ in the numerator of $\left[\frac{f(x)}{g(x)}\right]$ as N and the function $g(x)$in the denominator as D then, this rule can be written as

$$\frac{d}{dx}\left(\frac{N}{D}\right) = \frac{DN' - ND'}{D^2}$$

It should be noted that the products of one function and the derivative of the other function in numerator of the above formula involves subtraction unlike the product rule, where they are added. Therefore, the order in which the functions are chosen becomes very important- it has to be $DN' - ND'$.

We now give below a few examples from the present and the subsequent chapters where the quotient rule is used.

$$\frac{d}{dx}\frac{4x^3 + 3x^2 + 6}{2x^2 + 3}, \qquad \frac{d}{dx}\frac{\cot^{-1} x}{5x}, \qquad \frac{d}{dx}\frac{e^{-3x}}{1 + e^{2x}}, \qquad \frac{d}{dx}\left(\frac{3^x}{x^3}\right)$$

Study carefully the following Examples to learn how to use the quotient rule and then attempt the Practice Exercises given at the end of the Chapter.

Example 1: Find the following derivative with respect to x.

$$\frac{d}{dx}\frac{2x + 3}{3x^2 + 2}$$

Solution: We will apply the quotient rule to find $\frac{d}{dx}\frac{2x + 3}{3x^2 + 2}$. For this, let

$$f(x) = 2x + 3, \quad g(x) = 3x^2 + 2, \text{ so that } \frac{d}{dx}\frac{2x + 3}{3x^2 + 2} = \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

Applying the quotient rule, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{(3x^2 + 2)\left[\frac{d}{dx}(2x + 3)\right] - (2x + 3)\left[\frac{d}{dx}(3x^2 + 2)\right]}{(3x^2 + 2)^2}$$

$$= \frac{(3x^2 + 2)(2) - (2x + 3)(6x)}{9x^4 + 12x^2 + 4} = \frac{6x^2 + 4 - 12x^2 - 18x}{9x^4 + 12x^2 + 4} = \frac{-6x^2 - 18x + 4}{9x^4 + 12x^2 + 4}$$

Example 2: Find the following derivative with respect to x.

$$\frac{d}{dx}\frac{4x^3 + 3x^2 + 6}{2x^2 + 3}$$

Solution: We will apply the quotient rule to find $\frac{d}{dx}\frac{4x^3 + 3x^2 + 6}{2x^2 + 3}$. For this, let

$$f(x) = 4x^3 + 3x^2 + 6, \quad g(x) = 2x^2 + 3, \text{ so that } \frac{d}{dx}\frac{4x^3 + 3x^2 + 6}{2x^2 + 3} = \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

Applying the quotient rule, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

$$= \frac{(2x^2 + 3)\left[\frac{d}{dx}(4x^3 + 3x^2 + 6)\right] - (4x^3 + 3x^2 + 6)\left[\frac{d}{dx}(2x^2 + 3)\right]}{(2x^2 + 3)^2}$$

$$= \frac{(2x^2 + 3)(12x^2 + 6x) - (4x^3 + 3x^2 + 6)(4x)}{4x^4 + 12x^2 + 9}$$

$$= \frac{24x^4 + 12x^3 + 36x^2 + 18x - 16x^4 - 12x^3 - 24x}{4x^4 + 12x^2 + 9} = \frac{8x^4 + 36x^2 - 6x}{4x^4 + 12x^2 + 9}$$

Example 3: Find the following derivative with respect to x.

$$\frac{d}{dx}\frac{\sqrt{x}}{3x^2 + 2}$$

Solution: We will apply the quotient rule to find $\frac{d}{dx}\frac{\sqrt{x}}{3x^2 + 2}$. For this, let

$$f(x) = \sqrt{x}, \quad g(x) = 3x^2 + 2, \text{ so that}$$

$$\frac{d}{dx}\frac{\sqrt{x}}{3x^2 + 2} = \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

Applying the quotient rule, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{(3x^2 + 2)\left[\frac{d}{dx}(\sqrt{x})\right] - (\sqrt{x})\left[\frac{d}{dx}(3x^2 + 2)\right]}{(3x^2 + 2)^2}$$

$$= \frac{(3x^2 + 2)\left[\frac{d}{dx}(x^{1/2})\right] - (x^{1/2})\left[\frac{d}{dx}(3x^2 + 2)\right]}{(3x^2 + 2)^2} = \frac{(3x^2 + 2)\left(\frac{1}{2}x^{-1/2}\right) - (x^{1/2})(6x + 0)}{(3x^2 + 2)^2}$$

$$= \frac{\frac{3}{2}x^{3/2} + \frac{2}{2}x^{1/2} - 6x^{3/2}}{(9x^4 + 12x^2 + 4)} = \frac{-\frac{9}{2}x^{3/2} + x^{1/2}}{(9x^4 + 12x^2 + 4)}$$

$$= \frac{\frac{-9x^{3/2} + 2x^{1/2}}{2}}{(9x^4 + 12x^2 + 4)} = \frac{-9x^{3/2} + 2x^{1/2}}{2(9x^4 + 12x^2 + 4)} = - \frac{-9x\sqrt{x} + 2\sqrt{x}}{2(9x^4 + 12x^3 + 4x^2)}$$

Note that $x^2x^{-1/2} = x^{2+(-1/2)} = x^{2-1/2} = x^{\frac{4}{2}-\frac{1}{2}} = x^{\frac{4-1}{2}} = x^{\frac{3}{2}}$ and $xx^{1/2} = x^{1+\frac{1}{2}} = x^{\frac{2}{2}+\frac{1}{2}} = x^{\frac{2+1}{2}} = x^{\frac{3}{2}}$ (because fractions can be added or subtracted only when they have common denominator)

Chapter 4 Exercises

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d}{dx}\frac{8x^3 + 5x^2 + 6}{x + 2} =$$

2

$$\frac{d}{dx}\frac{x^5 + 1}{x - 1} =$$

3

$$\frac{d}{dx}\frac{x^2 + x - 2}{8x^8 + 5} =$$

Chapter 4 Solutions

1 We will apply the quotient rule to find $\frac{d}{dx}\frac{8x^3 + 5x^2 + 6}{x + 2}$. For this, let

$$f(x) = 8x^3 + 5x^2 + 6,\ g(x) = x + 2,\ \text{so that } \frac{d}{dx}\frac{8x^3 + 5x^2 + 6}{x + 2} = \frac{d}{dx}\left(\frac{f}{g}\right) = \ ?$$

$$\text{Applying the quotient rule, we have } \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

$$= \frac{(x + 2)\left[\frac{d}{dx}(8x^3 + 5x^2 + 6)\right] - (8x^3 + 5x^2 + 6)\left[\frac{d}{dx}(x + 2)\right]}{(x + 2)^2}$$

$$= \frac{(x + 2)(24x^2 + 10x) - (8x^3 + 5x^2 + 6)(1)}{(x^2 + 4x + 4)}$$

$$= \frac{24x^3 + 10x^2 + 48x^2 + 20x - 8x^3 - 5x^2 - 6}{(x^2 + 4x + 4)} = \frac{16x^3 + 53x^2 + 20x - 6}{(x^2 + 4x + 4)}$$

Explanation:

- Since the derivative of a constant is zero, we have $\frac{d}{dx}(2) = 0$.
- We will be using the identity $(a + b)^2 = a^2 + 2ab + b^2$ to find the square of the denominator, for example $(x + 2)^2 = (x^2 + 4x + 4)$.

We will apply the quotient rule to find $\frac{d}{dx}\frac{x^5 + 1}{x - 1}$. For this, let

$$f(x) = x^5 + 1,\quad g(x) = x - 1,\ \text{so that } \frac{d}{dx}\frac{x^5 + 1}{x - 1} = \frac{d}{dx}\left(\frac{f}{g}\right) = \ ?$$

Applying the quotient rule, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{(x - 1)\left[\frac{d}{dx}(x^5 + 1)\right] - (x^5 + 1)\left[\frac{d}{dx}(x - 1)\right]}{(x - 1)^2}$$

$$= \frac{(x - 1)(5x^4 + 0) - (x^5 + 1)(1 - 0)}{(x^2 - 2x + 1)} = \frac{(x - 1)(5x^4) - (x^5 + 1)(1)}{(x^2 - 2x + 1)}$$

$$= \frac{5x^5 - 5x^4 - (x^5 + 1)}{(x^2 - 2x + 1)} = \frac{5x^5 - 5x^4 - x^5 - 1}{(x^2 - 2x + 1)} = \frac{4x^5 - 5x^4 - 1}{(x^2 - 2x + 1)}$$

Explanation:

- Since the derivative of a constant is zero, we have $\frac{d}{dx}(1) = 0$.
- $\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1\ x^{1-1} = x^0 = 1.$

3 We will apply the quotient rule to find $\frac{d}{dx}\frac{x^2 + x - 2}{8x^8 + 5}$. For this, let

$f(x) = x^2 + x - 2, \quad g(x) = 8x^8 + 5$, so that $\frac{d}{dx}\frac{x^2 + x - 2}{8x^8 + 5} = \frac{d}{dx}\left(\frac{f}{g}\right) = ?$

Applying the quotient rule, we have $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$

$$= \frac{(8x^8 + 5)\left[\frac{d}{dx}(x^2 + x - 2)\right] - (x^2 + x - 2)\left[\frac{d}{dx}(8x^8 + 5)\right]}{(8x^8 + 5)^2}$$

$$= \frac{(8x^8 + 5)(2x + 1 - 0) - (x^2 + x - 2)(64x^7 + 0)}{(64x^{16} + 80x^8 + 25)}$$

$$= \frac{16x^9 + 10x + 8x^8 + 5 - (64x^9 + 64x^8 - 128x^7)}{(64x^{16} + 80x^8 + 25)}$$

$$= \frac{16x^9 + 10x + 8x^8 + 5 - 64x^9 - 64x^8 + 128x^7}{(64x^{16} + 80x^8 + 25)} = \frac{-48x^9 - 56x^8 + 128x^7 + 10x + 5}{(64x^{16} + 80x^8 + 25)}$$

5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

We start by giving below the derivatives of basic trigonometric functions.

$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \frac{d}{d\theta}\cos\theta = -\sin\theta$$

$$\frac{d}{d\theta}\tan\theta = \sec^2\theta \qquad \frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta$$

$$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta \qquad \frac{d}{d\theta}\cot\theta = -\csc^2\theta$$

Before proceeding further, let us recall the following important relations and identities:

$$\sin\theta = \frac{1}{\csc\theta} \text{ or } \csc\theta = \frac{1}{\sin\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \text{ or } \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

Example 1: Find the following derivative with respect to θ.

$$\frac{d}{d\theta}(3\sin 4\theta)$$

Solution: We will apply the chain rule to find $\frac{d}{d\theta}3\sin 4\theta$. Let $y = 3\sin 4\theta$

Put $u = 4\theta$, so that $y(u) = 3\sin u, \qquad u(\theta) = 4\theta, \qquad \frac{d}{d\theta}3\sin 4\theta = \frac{dy}{d\theta} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(3\sin u)\right]\left[\frac{d}{d\theta}(4\theta)\right] = \left[3\frac{d}{du}(\sin u)\right]\left[4\frac{d}{d\theta}(\theta)\right]$$

$$= [3\cos u]\,[4(1)] = (3\cos u)\,(4) = 12\cos u = 12\cos 4\theta$$

Example 2: Find the following derivative with respect to θ.

$$\frac{d}{d\theta}(\sin\theta + \tan\theta)$$

Solution: Apply the addition rule to get

$$\frac{d}{d\theta}(\sin\theta + \tan\theta) = \frac{d}{d\theta}\sin\theta + \frac{d}{d\theta}\tan\theta = \cos\theta + \sec^2\theta$$

Example 3: Find the following derivative with respect to θ.

$$\frac{d}{d\theta}\sqrt[3]{\sin\theta}$$

Solution: We will apply the chain rule to find $\frac{d}{d\theta}\sqrt[3]{\sin\theta}$. Let $y = \sqrt[3]{\sin\theta}$

Put $u = \sin\theta$, so that $y(u) = \sqrt[3]{u} = u^{1/3}, \qquad u(\theta) = \sin\theta, \qquad \frac{d}{d\theta}\sqrt[3]{\sin\theta} = \frac{dy}{d\theta} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(u^{1/3})\right]\left[\frac{d}{d\theta}(\sin\theta)\right] = \left(\frac{1}{3}u^{-2/3}\right)(\cos\theta)$$

$$= \left(\frac{1}{3}(\sin\theta)^{(-2/3)}\right)(\cos\theta) = \left(\frac{1}{3\sin^{2/3}\theta}\right)(\cos\theta) = \frac{\cos\theta}{3\sin^{2/3}\theta}$$

Example 4: Find the following derivative with respect to x.

$$\frac{d}{dx}3\cos\left(x^2 + 1\right)$$

Solution: We will apply the chain rule to find $\frac{d}{dx}3\cos\left(x^2+1\right)$. Let $y=3\cos\left(x^2+1\right)$

Put $u=x^2+1$, so that

$$y(u)=3\cos u, \qquad u(x)=x^2+1, \qquad \frac{d}{dx}3\cos\left(x^2+1\right)=\frac{dy}{dx}=\ ?$$

Applying the chain rule, we have

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\left[\frac{d}{du}(3\cos u)\right]\left[\frac{d}{dx}(x^2+1)\right]=\left[3\frac{d}{du}(\cos u)\right]\left[\frac{d}{dx}(x^2+1)\right]$$

$$=(-3\sin u)(2x+0)=(-3\sin u)(2x)=-6x\sin u=-6x\sin\left(x^2+1\right)$$

Chapter 5 Exercises

Set A

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d}{d\theta}5\tan\sqrt{\theta} =$$

2

$$\frac{d}{d\theta}\sqrt{1 + \sin\theta}$$

3

$$\frac{d}{d\theta}\sin(1 - \cos\theta) =$$

4

$$\frac{d}{dx}(3 + x\cos x) =$$

Set B

5

$$\frac{d}{dx}\sec x \csc x =$$

6 $$\frac{d}{d\theta}\cot\left(\frac{\theta}{\theta + 3}\right) =$$

7

$$\frac{d}{d\theta}\frac{1 + \cos\theta}{1 - \cos\theta} =$$

8

$$\frac{d}{d\theta}\sec^2(\theta^2 + 3) =$$

Chapter 5 Solutions

Set A

1 We will apply the chain rule to find $\frac{d}{d\theta}5\tan\sqrt{\theta}$. Let $y = 5\tan\sqrt{\theta}$

Put $u = \sqrt{\theta}$ so that $y(u) = 5\tan u$, $\frac{d}{d\theta}5\tan\sqrt{\theta} = \frac{dy}{d\theta} = ?$ Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(5\tan u)\right]\left[\frac{d}{d\theta}(\sqrt{\theta})\right] = \left[5\frac{d}{du}(\tan u)\right]\left[\frac{d}{d\theta}(\theta^{1/2})\right]$$

$$= (5\sec^2 u)\left(\frac{1}{2}\theta^{-1/2}\right) = (5\sec^2\sqrt{\theta})\left(\frac{1}{2\sqrt{\theta}}\right) = \frac{5\sec^2\sqrt{\theta}}{2\sqrt{\theta}} = \frac{(5\sec^2\sqrt{\theta})}{2\sqrt{\theta}}\frac{\sqrt{\theta}}{\sqrt{\theta}} = \frac{5\sqrt{\theta}\,\sec^2\sqrt{\theta}}{2\theta}$$

$$= \frac{5\sqrt{\theta}}{2\theta\cos^2\theta}$$

Explanation:

- We have used constant multiple rule to get $\frac{d}{du}(5\tan u) = 5\frac{d}{du}(\tan u)$.
- Note that $\sqrt{\theta} = \theta^{1/2}$, $\theta^{-1/2} = \frac{1}{\theta^{1/2}}$ and $\sqrt{\theta}\sqrt{\theta} = \sqrt{\theta^2} = \theta$.
- $\frac{5\sec^2\sqrt{\theta}}{2\sqrt{\theta}}$, $\frac{5\sqrt{\theta}\,\sec^2\sqrt{\theta}}{2\theta}$ and $\frac{5\sqrt{\theta}}{2\theta\cos^2\theta}$ are all correct answers. However, only the last two have rational denominators.

We will apply the chain rule to find $\frac{d}{d\theta}\sqrt{1 + \sin\theta}$. Let $y = \sqrt{1 + \sin\theta}$

Put $u = 1 + \sin\theta$, so that

$y(u) = \sqrt{u}$, $u(\theta) = 1 + \sin\theta$, $\frac{d}{d\theta}\sqrt{1 + \sin\theta} = \frac{dy}{d\theta} = ?$

Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(\sqrt{u})\right]\left[\frac{d}{d\theta}(1 + \sin\theta)\right] = \left[\frac{d}{du}(u^{1/2})\right]\left[\frac{d}{d\theta}(1 + \sin\theta)\right]$$

$$= \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(0 + \cos\theta) = \left(\frac{1}{2u^{1/2}}\right)(\cos\theta) = \frac{\cos\theta}{2u^{1/2}} = \frac{\cos\theta}{2\sqrt{1 + \sin\theta}}$$

e above answer is correct. However, if we wish, we can rationalize the denominator by multiplying with $\frac{\sqrt{1 + \sin\theta}}{\sqrt{1 + \sin\theta}}$ as follows.

$$\frac{dy}{d\theta} = \frac{\cos\theta \qquad \sqrt{1 + \sin\theta}}{2\sqrt{1 + \sin\theta}\sqrt{1 + \sin\theta}} = \frac{\cos\theta\sqrt{1 + \sin\theta}}{2(1 + \sin\theta)}$$

Explanation: $\sqrt{u} = u^{1/2}$.

3

We will apply the chain rule to find $\frac{d}{d\theta}\sin(1 - \cos\theta)$. Let $y = \sin(1 - \cos\theta)$

Put $u = 1 - \cos\theta$, so that $y(u) = \sin u,\ u(\theta) = 1 - \cos\theta,\quad \frac{dy}{d\theta} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}\sin u\right]\left[\frac{d}{d\theta}(1 - \cos\theta)\right] = (\cos u)\ [0 - (-\sin\theta)]$$

$$= [\cos(1 - \cos\theta)]\ [\sin\theta] = \sin\theta\cos\mathbf{(}1 - \cos\theta\mathbf{)}$$

Explanation:

- Note that $\sin(1 - \cos\theta)$ does not imply multiplication of two functions. Instead, $(1 - \cos\theta)$ is the argument of sine function.
- Note that $\frac{d}{dx}(1) = 0$ (because the derivative of a constant is zero).

4

Applying the addition rule, we have

$$\frac{d}{dx}(3 + x\cos x) = \frac{d}{dx}(3) + \frac{d}{dx}(x\cos x) = 0 + \frac{d}{dx}(x\cos x) = \frac{d}{dx}(x\cos x)$$

To find $\frac{d}{dx}(x\cos x)$, we need to apply the product rule. For this, let

$f(x) = x,\ \ g(x) = \cos x$ so that $\frac{d}{dx}(x\cos x) = \frac{d}{dx}(fg) = ?$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = x\left[\frac{d}{dx}(\cos x)\right] + \cos x\left[\frac{d}{dx}(x)\right]$$

$$= x(-\sin x) + \ \cos x(1) = -x\sin x + \ \cos x$$

Substituting the value of $\frac{d}{dx}(fg)\left(= \frac{d}{dx}(x\cos x)\right)$, we have

$$\frac{d}{dx}(3 + x\cos x) = \frac{d}{dx}(x\cos x) = -x\sin x + \ \cos x$$

Explanation: Note that $\frac{d}{dx}(3) = 0$ (because the derivative of a constant is zero).

Set B

5 We will apply the product rule to find $\frac{d}{dx}\sec x\csc x$. For this, let

$$f(x) = \sec x, \quad g(x) = \csc x, \text{ so that } \frac{d}{dx}\sec x\csc x = \frac{d}{dx}(fg) = \ ?$$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = (\sec x)\left[\frac{d}{dx}(\csc x)\right] + (\csc x)\left[\frac{d}{dx}(\sec x)\right]$$

$$= (\sec x)(-\csc x\cot x) + (\csc x)(\sec x\tan x) = \sec x\csc x\mathbf{(}-\cot x + \tan x\mathbf{)}$$

The above answer is correct. However, if we wish, we can simplify as follows.

$$\frac{d}{dx}(fg) = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right)\left(-\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}\right) = \frac{-1}{\cos x\sin x}\frac{\cos x}{\sin x} + \frac{1}{\cos x\sin x}\frac{\sin x}{\cos x} = \frac{-1}{\sin^2 x} + \frac{1}{\cos^2 x} = -\csc^2 x + \sec^2 x$$

Explanation:

- We have used the relations $\sec x = \frac{1}{\cos x}, \ \csc x = \frac{1}{\sin x}, \ \cot x = \frac{\cos x}{\sin x}, \ \tan x = \frac{\sin x}{\cos x}$.
- $\sec x\csc x(-\cot x + \tan x), \ \frac{-1}{\sin^2 x} + \frac{1}{\cos^2 x}$ and $-\csc^2 x + \sec^2 x$ are all correct answers.

6 We will apply the chain rule to find $\frac{d}{d\theta}\cot\left(\frac{\theta}{\theta+3}\right)$. Let $y = \cot\left(\frac{\theta}{\theta+3}\right)$

$$\text{Put } u = \frac{\theta}{\theta+3}, \text{ so that } y(u) = \cot u, \quad u(\theta) = \frac{\theta}{\theta+3}, \quad \frac{d}{d\theta}\cot\left(\frac{\theta}{\theta+3}\right) = \frac{dy}{d\theta} = \ ?$$

Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(\cot u)\right]\left[\frac{d}{d\theta}\left(\frac{\theta}{\theta+3}\right)\right] = (-\csc^2 u)\left[\frac{d}{d\theta}\frac{\theta}{\theta+3}\right]$$

We need to apply the quotient rule to find $\frac{d}{d\theta}\left(\frac{\theta}{\theta+3}\right)$. For this, let

$$f(\theta) = \theta, \quad g(\theta) = \theta + 3, \text{ so that } \frac{d}{d\theta}\left(\frac{\theta}{\theta+3}\right) = \frac{d}{d\theta}\left(\frac{f}{g}\right) = \ ?$$

Applying the quotient rule, we have

$$\frac{d}{d\theta}\left(\frac{\theta}{\theta+3}\right) = \frac{d}{d\theta}\left(\frac{f}{g}\right) = \frac{g\frac{df}{d\theta} - f\frac{dg}{d\theta}}{g^2} = \frac{(\theta+3)\left[\frac{d}{d\theta}(\theta)\right] - \theta\left[\frac{d}{d\theta}(\theta+3)\right]}{(\theta+3)^2}$$

$$= \frac{(\theta + 3)(1) - \theta(1 + 0)}{(\theta^2 + 6\theta + 9)} = \frac{(\theta + 3) - (\theta)}{(\theta^2 + 6\theta + 9)} = \frac{\theta + 3 - \theta}{(\theta^2 + 6\theta + 9)} = \frac{3}{(\theta^2 + 6\theta + 9)}$$

Substituting the above value of $\frac{d}{d\theta}\left(\frac{\theta}{\theta + 3}\right)$ in the expression for $\frac{dy}{d\theta}$, we get

$$\frac{dy}{d\theta} = (- \csc^2 u)\left[\frac{3}{(\theta^2 + 6\theta + 9)}\right] = \frac{- 3\csc^2\left(\frac{\theta}{\theta + 3}\right)}{\left(\theta^2 + 6\theta + 9\right)} = \frac{- 3}{\left(\theta^2 + 6\theta + 9\right)\sin^2\left(\frac{\theta}{\theta + 3}\right)}$$

Explanation: Note that $(\theta + 3)^2 = \theta^2 + 6\theta + 9$ because $(a + b)^2 = a^2 + 2ab + b^2$.

Also, we have $\csc\left(\frac{\theta}{\theta + 3}\right) = \frac{1}{\sin\left(\frac{\theta}{\theta + 3}\right)}$.

We will apply the quotient rule to find $\frac{d}{d\theta}\frac{1 + \cos\theta}{1 - \cos\theta}$. For this, let

$f(\theta) = 1 + \cos\theta, \quad g(\theta) = 1 - \cos\theta$, so that $\frac{d}{d\theta}\frac{1 + \cos\theta}{1 - \cos\theta} = \frac{d}{d\theta}\left(\frac{f}{g}\right) = \ ?$

Applying the quotient rule, we have $\frac{d}{d\theta}\left(\frac{f}{g}\right) = \frac{g\frac{df}{d\theta} - f\frac{dg}{d\theta}}{g^2}$

$$= \frac{(1 - \cos\theta)\left[\frac{d}{d\theta}(1 + \cos\theta)\right] - (1 + \cos\theta)\left[\frac{d}{d\theta}(1 - \cos\theta)\right]}{(1 - \cos\theta)^2}$$

$$= \frac{(1 - \cos\theta)(- \sin\theta) - (1 + \cos\theta)[- (- \sin\theta)]}{\left(1 - 2\cos\theta + \cos^2\theta\right)}$$

$$= \frac{- \sin\theta + \cos\theta\sin\theta - \sin\theta - \cos\theta\sin\theta}{\left(1 - 2\cos\theta + \cos^2\theta\right)} = \frac{- 2\sin\theta}{\left(1 - 2\cos\theta + \cos^2\theta\right)}$$

Explanation: Note that $\frac{d}{d\theta}(1) = 0$ and $-(- \sin\theta) = \sin\theta$.

8 We will apply the chain rule to find $\frac{d}{d\theta}\sec^2(\theta^2 + 3)$. Let $y = \sec^2(\theta^2 + 3)$

Put $u = \sec(\theta^2 + 3)$, so that $y(u) = u^2, \ u(\theta) = \sec(\theta^2 + 3), \ \frac{dy}{d\theta} = \ ?$

Apply the chain rule: $\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(u^2)\right]\left[\frac{d}{dx}\sec(\theta^2 + 3)\right]$

$$= (2u)\left[\frac{d}{d\theta}\sec\left(\theta^2 + 3\right)\right] = 2\sec\left(\theta^2 + 3\right)\left[\frac{d}{d\theta}\sec\left(\theta^2 + 3\right)\right]$$

We need to apply the chain rule once again to find $\frac{d}{d\theta}\sec\left(\theta^2 + 3\right)$. For this, let $z = \sec\left(\theta^2 + 3\right)$. Put $v = \theta^2 + 3$, so that $z(v) = \sec v, \quad v(\theta) = \theta^2 + 3, \ \frac{dz}{d\theta} = \ ?$

Applying the chain rule, we have

$$\frac{dz}{d\theta} = \frac{dz}{dv}\frac{dv}{d\theta} = \frac{d}{dv}(\sec v)\,\frac{dv}{d\theta} = (\sec v \tan v)\left[\frac{d}{d\theta}\left(\theta^2 + 3\right)\right]$$

$$= \sec v \tan v(2\theta + 0) = \sec v \tan v(2\theta) = 2\theta \sec\left(\theta^2 + 3\right)\tan\left(\theta^2 + 3\right)$$

Substituting the above value of $\frac{dz}{d\theta}\left[= \frac{d}{d\theta}\sec\left(\theta^2 + 3\right)\right]$ in the expression for $\frac{dy}{d\theta}$ we get

$$\frac{dy}{d\theta} = 2\sec\left(\theta^2 + 3\right)\left[2\theta\sec\left(\theta^2 + 3\right)\tan\left(\theta^2 + 3\right)\right] = 4\theta\sec^2\left(\theta^2 + 3\right)\tan\left(\theta^2 + 3\right)$$

6 DERIVATIVES OF INVERSE TRIGO FUNCTIONS

The derivatives of inverse trigonometric functions are:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \text{ where } |x| < 1, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \text{ where } |x| < 1$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \text{ where } |x| > 1, \quad \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}} \text{ where } |x| > 1$$

Example 1: Find the following derivative with respect to x.

$$\frac{d}{dx}4\tan^{-1}3x$$

Solution: We will apply the chain rule to find $\frac{d}{dx}4\tan^{-1}3x$. Let $y = 4\tan^{-1}3x$

Put $u = 3x$, so that $y(u) = 4\tan^{-1}u \qquad u(x) = 3x, \qquad \frac{d}{dx}4\tan^{-1}3x = \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(4\tan^{-1}3x)\right]\left[\frac{d}{dx}(3x)\right] = \left[4\frac{d}{du}(\tan^{-1}3x)\right]\left[\frac{d}{dx}(3x)\right]$$

$$= 4\left(\frac{1}{1+u^2}\right)(3) = \frac{12}{1+(3x)^2} = \frac{12}{1+9x^2}$$

Example 2: Find the following derivative with respect to x.

$$\frac{d}{dx}\tan^{-1}x\cot^{-1}x$$

Solution: We will apply the product rule to find $\frac{d}{dx}\tan^{-1}x\cot^{-1}x$. For this, let

$$f(x) = \tan^{-1}x \quad g(x) = \cot^{-1}x, \text{ so that } \frac{d}{dx}\tan^{-1}x\cot^{-1}x = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = \tan^{-1}x\left[\frac{d}{dx}\cot^{-1}x\right] + \cot^{-1}x\left[\frac{d}{dx}\tan^{-1}x\right]$$

$$= \tan^{-1}x\left(-\frac{1}{1+x^2}\right) + \cot^{-1}x\left(\frac{1}{1+x^2}\right) = -\tan^{-1}x\left(\frac{1}{1+x^2}\right) + \cot^{-1}x\left(\frac{1}{1+x^2}\right)$$

$$= (-\tan^{-1}x + \cot^{-1}x)\left(\frac{1}{1+x^2}\right) = \left(\frac{1}{1+x^2}\right)(-\tan^{-1}x + \cot^{-1}x)$$

Chapter 6 Exercises

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d}{dx}4\sin^{-1}x^3 =$$

2

$$\frac{d}{dx}(\cos^{-1}x^4)^4 =$$

3

$$\frac{d}{dx}(2+x)\csc^{-1}x =$$

4

$$\frac{d}{dx}\left(\frac{\sin^{-1}x}{\cos x}\right) =$$

Chapter 6 Solutions

1 We will apply the chain rule to find $\frac{d}{dx}4\sin^{-1}x^3$. Let $y = 4\sin^{-1}x^3$

Put $u = x^3$, so that $y(u) = 4\sin^{-1}u, \qquad u(x) = x^3, \qquad \frac{d}{dx}4\sin^{-1}x^3 = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}\left(4\sin^{-1}u\right)\right]\left[\frac{d}{dx}\left(x^3\right)\right] = \left[4\frac{d}{du}\left(\sin^{-1}u\right)\right]\left[\frac{d}{dx}\left(x^3\right)\right]$$

$$= 4\left(\frac{1}{\sqrt{1 - u^2}}\right)\left(3x^2\right) = \frac{12x^2}{\sqrt{1 - \left(x^3\right)^2}} = \frac{12x^2}{\sqrt{1 - x^6}}$$

Explanation:

- We have used constant multiple rule to get $\frac{d}{du}\left(4\sin^{-1}u\right) = 4\frac{d}{du}\left(\sin^{-1}u\right)$.

We will apply the chain rule to find $\frac{d}{dx}\left(\cos^{-1}x^4\right)^4$. Let $y = \left(\cos^{-1}x^4\right)^4$

Put $u = \cos^{-1}x^4$, so that $y(u) = u^4$, $u(x) = \cos^{-1}x^4$, $\frac{d}{dx}\left(\cos^{-1}x^4\right)^4 = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$= \left[\frac{d}{du}\left(u^4\right)\right]\left[\frac{d}{dx}\cos^{-1}x^4\right] = \left(4u^3\right)\left[\cos^{-1}x^4\right] = 4\left(\cos^{-1}x^4\right)^3\left[\frac{d}{dx}\cos^{-1}x^4\right]$$

We need to apply the chain rule once again to find $\frac{d}{dx}\cos^{-1}x^4$. Let $z = \cos^{-1}x^4$

Put $v = x^4$, so that $z(v) = \cos^{-1}v, \qquad v(x) = x^4, \qquad \frac{d}{dx}\cos^{-1}x^4 = \frac{dz}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dz}{dx} = \frac{dz}{dv}\frac{dv}{dx} = \left[\frac{d}{dv}\left(\cos^{-1}v\right)\right]\left[\frac{d}{dx}\left(x^4\right)\right] = \left(-\frac{1}{\sqrt{1 - v^2}}\right)\left(4x^3\right) = -\frac{4x^3}{\sqrt{1 - x^8}}$$

Substituting the above value of $\frac{d}{dx}\cos^{-1}v\left[=\frac{dz}{dx}\right]$ in the expression for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = 4\left(\cos^{-1}v\right)^3\left[-\frac{4x^3}{\sqrt{1 - x^8}}\right] = -\frac{16x^3\left(\cos^{-1}x^4\right)^3}{\sqrt{1 - x^8}}$$

Explanation: $\left(x^4\right)^2 = x^8$ since $\left(x^m\right)^n = x^{mn}$.

We will apply the chain rule to find $\frac{d}{dx}(2+x)\csc^{-1}x$. For this, let

$$f(x) = 2 + x\,,\quad g(x) = \csc^{-1}x,\quad \text{so that}\ \frac{d}{dx}(2+x)\csc^{-1}x = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$= (2+x)\left[\frac{d}{dx}(\csc^{-1}x)\right] + (\csc^{-1}x)\left[\frac{d}{dx}(2+x)\right]$$

$$= (2+x)\left[-\frac{1}{|x|\sqrt{x^2-1}}\right] + (\csc^{-1}x)\left[\frac{d}{dx}(2+x)\right]$$

$$= -\frac{(2+x)}{|x|\sqrt{x^2-1}} + (\csc^{-1}x)(1) = -\frac{(2+x)}{|x|\sqrt{x^2-1}} + \csc^{-1}x \text{ where } |x| > 1$$

Explanation:

- We have $\frac{d}{dx}(2+x) = \frac{d}{dx}(2) + \frac{d}{dx}(x) = 0 + 1 = 1$.
- Note that the function $y = \csc^{-1}x$ is defined for $|x| > 1$, where the slope of the graph of $y = \csc^{-1}x$ (which equals the derivative of $\csc^{-1}x$) is always negative. This explains the presence of absolute value $|x|$ in $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$.

4 We will quotient the chain rule to find $\frac{d}{dx}\left(\frac{\sin^{-1}x}{\cos x}\right)$. For this, let

$$f(x) = \sin^{-1}x,\qquad g(x) = \cos x,\quad \text{so that}\ \frac{d}{dx}\left(\frac{\sin^{-1}x}{\cos x}\right) = \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

Applying the quotient rule, we have $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$

$$= \frac{\cos x\frac{d}{dx}(\sin^{-1}x) - \sin^{-1}x\frac{d}{dx}(\cos x)}{(\cos x)^2} = \frac{\cos x\left(\frac{1}{\sqrt{1-x^2}}\right) - \sin^{-1}x(-\sin x)}{\cos^2 x}$$

$$= \frac{\frac{\cos x}{\sqrt{1-x^2}} - (-\sin^{-1}x\sin x)}{\cos^2 x} = \frac{\frac{\cos x - \sqrt{1-x^2}(-\sin^{-1}x\sin x)}{\sqrt{1-x^2}}}{\cos^2 x}$$

$$= \frac{\cos x + \sqrt{1 - x^2}\sin^{-1} x \sin x}{\sqrt{1 - x^2}\cos^2 x}$$

7 DERIVATIVES OF EXPONENTIAL FUNCTIONS

In this chapter, we will be dealing with derivative of exponential function of the form $f(x) = e^x$, where e is known as Euler's number. e is an irrational number, so it does not have exact value. Its value to ten decimal places is 2.7118281828. Some important properties of exponential functions are given below.

$$e^{x+y} = e^x e^y, \qquad e^{-x} = \frac{1}{e^x}, \qquad e^{x-y} = e^x e^{-y} = \frac{e^x}{e^y}, \qquad (e^x)^a = e^{ax}, \qquad e^0 = 1$$

As regards the derivative of exponential function, this function has a unique property- it equals its own derivative:

$$\frac{d}{dx}e^x = e^x$$

Another important result that follows from the previous one simply by use of the chain rule (see Example1) is

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

Example 1: Find the following derivative with respect to x.

$$\frac{d}{dx}e^{ax}$$

Solution: We will apply the chain rule to find $\frac{d}{dx}e^{ax}$. Let $y = e^{ax}$

Put $u = ax$, so that $y(u) = e^u$, $\qquad u(x) = ax, \qquad \frac{d}{dx}e^{ax} = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(e^u)\right]\left[\frac{d}{dx}(ax)\right] = \left[\frac{d}{du}(e^u)\right](a) = (e^u)(a) = ae^{ax}$$

Example 2: Find the following derivative with respect to x.

$$\frac{d}{dx}4e^{(5x^2+6)}$$

Solution: We will apply the chain rule to find $\frac{d}{dx}4e^{(5x^2+6)}$. Let $y = 4e^{(5x^2+6)}$

Put $u = 5x^2 + 6$, so that

$$y(u) = 4e^{u}, \qquad u(x) = 5x^2 + 6, \qquad \frac{d}{dx}4e^{(5x^2+6)} = \frac{dy}{dx} = \ ?$$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(4e^{u})\right]\left[\frac{d}{dx}(5x^2 + 6)\right] = \left[4\frac{d}{du}(e^{u})\right](10x + 0)$$

$$= (4e^{u})(10x) = \left[4e^{(5x^2+6)}\right](10x) = 40xe^{(5x^2+6)}$$

Chapter 7 Exercises

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d}{dx}\sqrt{3 + e^{-2x}} =$$

2

$$\frac{d}{dx}\left(4e^{4x^4} + 3e^{3x^3} + 7\right) =$$

3

$$\frac{d}{dx}(x^5 e^x) =$$

4

$$\frac{d}{dx}\left(\frac{e^x}{1 + x^2}\right) =$$

Chapter 7 Solutions

We will apply the chain rule to find $\frac{d}{dx}\sqrt{3 + e^{-2x}}$. Let $y = \sqrt{3 + e^{-2x}}$

Put $u = 3 + e^{-2x}$, so that

$y(u) = \sqrt{u}, \quad u(x) = 3 + e^{-2x}, \quad \frac{d}{dx}\sqrt{3 + e^{-2x}} = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(\sqrt{u})\right]\left[\frac{d}{dx}(3 + e^{-2x})\right] = \left[\frac{d}{du}(u^{1/2})\right]\left[\frac{d}{dx}(3 + e^{-2x})\right]$$

$$= \left[\frac{1}{2}u^{-1/2}\right](0 - 2e^{-2x}) = \left[\frac{1}{2u^{1/2}}\right](-2e^{-2x}) = \frac{-e^{-2x}}{\sqrt{u}} = -\frac{e^{-2x}}{\sqrt{3 + e^{-2x}}}$$

The above answer is correct. However, if we wish, we can rationalize the denominator as follows.

$$\frac{dy}{dx} = \frac{-e^{-2x}\sqrt{3 + e^{-2x}}}{\sqrt{3 + e^{-2x}}\sqrt{3 + e^{-2x}}} = \frac{-e^{-2x}\sqrt{3 + e^{-2x}}}{(3 + e^{-2x})}$$

Explanation:

- Note that $\sqrt{u} = u^{1/2}$, $u^{-1/2} = \frac{1}{u^{1/2}}$.
- Because $\frac{d}{dx}(e^{ax}) = ae^{ax}$, we have $\frac{d}{dx}(e^{-2x}) = -2e^{-2x}$ where $a = -2$.
- $\sqrt{3 + e^{-2x}}\sqrt{3 + e^{-2x}} = \sqrt{(3 + e^{-2x})^2} = (3 + e^{-2x})$.

2 Applying the addition rule, we have

$$\frac{d}{dx}\left(4e^{4x^4} + 3e^{3x^3} + 7\right) = \frac{d}{dx}\left(4e^{4x^4}\right) + \frac{d}{dx}\left(3e^{3x^3}\right) + \frac{d}{dx}(7)$$

$$= \frac{d}{dx}\left(4e^{4x^4}\right) + \frac{d}{dx}\left(3e^{3x^3}\right) + 0 = \frac{d}{dx}\left(4e^{4x^4}\right) + \frac{d}{dx}\left(3e^{3x^3}\right)$$

We will apply the chain rule to find $\frac{d}{dx}\left(4e^{4x^4}\right)$. Let $y = 4e^{4x^4}$

Put $u = 4x^4$, so that $y(u) = 4e^{u}, \qquad u(x) = 4x^4, \qquad \frac{d}{dx}4e^{4x^4} = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(4e^u)\frac{du}{dx} = \left[4\frac{d}{du}(e^u)\right]\left[\frac{d}{dx}(4x^4)\right] = (4e^u)(16x^3) = 64x^3e^{4x^4}$$

We will apply the chain rule for a second time to find $\frac{d}{dx}\left(3e^{3x^3}\right)$. Let $z = 3e^{3x^3}$

Put $v = 3x^3$, so that $z(v) = 3e^v, \quad v(x) = 3x^3, \quad \frac{d}{dx}3e^{3x^3} = \frac{dz}{dx} = ?$

Applying the chain rule, we have

$$\frac{dz}{dx} = \frac{dz}{dv}\frac{dv}{dx} = \frac{d}{dv}(3e^v)\frac{dv}{dx} = \left[3\frac{d}{dv}(e^v)\right]\left[\frac{d}{dx}(3x^3)\right] = (3e^v)(9x^2) = 27x^2e^{3x^3}$$

Substituting the above value of $\frac{d}{dx}4e^{4x^4}\left[=\frac{dy}{dx}\right]$ and $\frac{d}{dx}3e^{3x^3}\left[=\frac{dz}{dx}\right]$, we get

$$\frac{d}{dx}\left(4e^{4x^4} + 3e^{3x^3} + 7\right) = 64x^3e^{4x^4} + 27x^2e^{3x^3} = x^2(64xe^{4x^4} + 27e^{3x^3})$$

3 We will apply the product rule to find $\frac{d}{dx}(x^5e^x)$. For this, let

$$f(x) = x^5,\ g(x) = e^x,\ \text{so that } \frac{d}{dx}(x^5e^x) = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = x^5\left[\frac{d}{dx}e^x\right] + e^x\left[\frac{d}{dx}x^5\right]$$

$$= x^5(e^x) + e^x(5x^4) = x^5e^x + 5x^4e^x = x^4e^x(x+5)$$

Explanation:

- Factoring out x^4e^x, we have $x^5e^x + 5x^4e^x = x^4e^x(x+5)$.

4 We will apply the quotient rule to find $\frac{d}{dx}\left(\frac{e^x}{1+x^2}\right)$. For this, let

$$f(x) = e^x, \quad g(x) = 1 + x^2, \quad \text{so that } \frac{d}{dx}\left(\frac{e^x}{1+x^2}\right) = \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

Applying the quotient rule, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{(1+x^2)\left[\frac{d}{dx}e^x\right] - e^x\left[\frac{d}{dx}(1+x^2)\right]}{(1+x^2)^2}$$

$$= \frac{(1 + x^2)(e^x) - e^x(0 + 2x)}{(1 + 2x^2 + x^4)} = \frac{e^x(1 + x^2) - (e^x)(2x)}{(1 + 2x^2 + x^4)} = \frac{e^x\left(1 + x^2 - 2x\right)}{\left(1 + 2x^2 + x^4\right)}$$

8 DERIVATIVES OF LOGARITHMIC FUNCTIONS

We start by recalling that logarithmic function to any base $b > 1,\ b \neq 1$, is defined as

$$y = \log_b x \text{ if } b^y = x$$

That is, y is the power to which b must be raised to get x. If the base is Euler's number $e\ (e \approx 2.7118281828)$, then we say that it is natural logarithm function and denote it as $\ln x$. In other words, $\ln x = \log_e x$ and

$$y = \ln x \qquad \text{is equivalent to} \qquad e^y = x.$$

The derivative of logarithmic function is

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

An important result that follows from the previous one simply by use of the chain rule (see Example1) is

$$\frac{d}{dx}\ln ax = \frac{1}{x}$$

It is interesting to note that a has disappeared on the right hand side of the above formula.
The following result, known as the change of base formula, allows us to express the general logarithm to any base b in terms of natural logarithms.

$$\log_b x = \frac{\ln x}{\ln b}$$

Since $(\ln b)$ is a constant, the above formula, combined with the constant multiple rule allows us to find the derivative of the logarithm to any base b:

$$\frac{d}{dx}\log_b x = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}\ln x = \frac{1}{\ln bx} = \frac{1}{x\ln b}$$

Note also the derivative of the power function b^x:

$$\frac{d}{dx}b^x = b^x \ln b$$

Before proceeding further, we state below the important properties of logarithms.

$$\ln(xy) = \ln x + \ln y, \qquad \ln\frac{x}{y} = \ln x - \ln y, \qquad \ln x^n = n\ln x$$

$$\ln(e^x) = x, \qquad \ln e = 1, \qquad \ln 1 = 0$$

$e^{\ln x}$

$= x,$

$\ln x$

$\ln\frac{1}{x} = \ln x^{-1} =-$

Example 1: Find the following derivative with respect to x.

$$\frac{d}{dx}\ln ax$$

Solution: We will apply the chain rule to find $\frac{d}{dx}\ln ax$. Let $y = \ln ax$

$$\text{Put } u = ax, \text{ so that } y(u) = \ln u, \qquad u(x) = ax, \qquad \frac{d}{dx}\ln ax = \frac{dy}{dx} = \ ?$$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(\ln u)\right]\left[\frac{d}{dx}(ax)\right] = \left[\frac{d}{du}(\ln u)\right]\left[\frac{d}{dx}(ax)\right]$$

$$= \left[\frac{1}{u}\right][a] = \left(\frac{1}{ax}\right)(a) = \frac{1}{x}$$

Example 2: Find the following derivative with respect to x.

$$\frac{d}{dx}x^2\ln x$$

Solution: We will apply the product rule to find $\frac{d}{dx}x^2\ln x$. For this, let

$$f(x) = x^2, \ g(x) = \ln x, \text{ so that } \frac{d}{dx}x^2\ln x = \frac{d}{dx}(fg) = \ ?$$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = (x^2)\left[\frac{d}{dx}\ln x\right] + \ln x\left[\frac{d}{dx}(x^2)\right]$$

$$= (x^2)\left(\frac{1}{x}\right) + \ln x(2x) = x + 2x\ln x = x(1 + 2\ln x)$$

Example 3: Find the following derivative with respect to x.

$$\frac{d}{dx}\log_3 x$$

Solution: Apply the change of base formula $\log_b x = \frac{\ln x}{\ln b}$, we get $\log_3 x = \frac{\ln x}{\ln 3}$.

$$\frac{d}{dx}(\log_3 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 3}\right) = \frac{1}{\ln 3}\left[\frac{d}{dx}\ln x\right] = \frac{1}{\ln 3}\left(\frac{1}{x}\right) = \frac{1}{x\ln 3}$$

Example 4: Find the following derivative with respect to x.

$$\frac{d\ 3^{x}}{dx_{x^{3}}}$$

Solution: We will apply the quotient rule to find $\frac{d\ 3^{x}}{dx_{x^{3}}}$. For this, let

$f(x) = 3^x, \qquad g(x) = x^3, \quad \text{so that } \frac{d\ 3^{x}}{dx_{x^{3}}} = \frac{d}{dx}\left(\frac{f}{g}\right) = \ ?$

Applying the quotient rule, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{x^3\left[\frac{d}{dx}(3^x)\right] - 3^x\left[\frac{d}{dx}(x^3)\right]}{(x^3)^2}$$

To find $\frac{d}{dx}(3^x)$, recall the formula $\frac{d}{dx}(b^x) = b^x \ln b$ and note that $b = 3.$

So, $\frac{d}{dx}(3^x) = 3^x \ln 3.$

Substituting this value, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{x^3(3^x \ln 3) - 3^x(3x^2)}{(x^3)^2} = \frac{x^3 3^x \ln 3 - 3x^2 3^x}{x^6}$$

$$= \frac{x^2 3^x(x\ln 3 - 3)}{x^6} = \frac{3^x(x \ln 3)}{x^4}$$

Example 5: Find the following derivative with respect to x.

$$\frac{d}{dx}\ln|\cos x|$$

Solution: We will apply the chain rule to find $\frac{d}{dx}\ln|\cos x|$. Let $y = \ln|\cos x|$

Put $u = \cos x$, so that $y(u) = \ln u$, $u(x) = \cos x$, $\frac{d}{dx}\ln|\cos x| = \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}\ln u\right]\left[\frac{d}{dx}\cos x\right] = \left(\frac{1}{u}\right)(-\sin x) = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x$$

Chapter 8 Exercises

Set A

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d}{dx}e^{x}(\ln x + x) =$$

2

$$\frac{d}{dx}\left(\frac{\ln x}{x^3}\right) =$$

3

$$\frac{d}{dx}\ln(\ln x) =$$

Set B

4

$$\frac{d}{dx}\ln(x^2 + 5) =$$

5

$$\frac{d}{d\theta}\ln|\sin\theta| =$$

6

$$\frac{d}{dx}\log_7(2x + 5) =$$

Set C

7

$$\frac{d}{dx}5^{x^2} =$$

8

$$\frac{d}{dx}\ln\sqrt{x + 3} =$$

$$\frac{d}{dx}\ln(\tan x + \sec x) =$$

Chapter 8 Solutions

Set A

1 We will apply the product rule to find $\frac{d}{dx}e^x(\ln x + x)$. For this, let

$$f(x) = e^x,\ g(x) = \ln x + x\ ,\ \text{ so that} \frac{d}{dx}e^x(\ln x + x) = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$= e^x\left[\frac{d}{dx}(\ln x + x)\right] + (\ln x + x)\left[\frac{d}{dx}e^x\right] = e^x\left(\frac{1}{x} + 1\right) + (\ln x + x)\left(e^x\right)$$

$$= e^x\left(\frac{1}{x} + 1\right) + e^x(\ln x + x) = e^x\left(\frac{1}{x} + 1 + \ln x + x\right)$$

2 We will apply the quotient rule to find $\frac{d}{dx}\left(\frac{\ln x}{x^3}\right)$. For this, let

$$f(x) = \ln x, \qquad g(x) = x^3, \quad \text{so that } \frac{d}{dx}\left(\frac{\ln x}{x^3}\right) = \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

Applying the quotient rule, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{x^3\left[\frac{d}{dx}\ln x\right] - \ln x\left[\frac{d}{dx}(x^3)\right]}{(x^3)^2} = \frac{x^3\left(\frac{1}{x}\right) - \ln x\left(3x^2\right)}{x^6}$$

$$= \frac{x^2 - 3x^2\ln x}{x^6} = \frac{x^2(1 - 3\ln x)}{x^6} = \frac{1 - 3\ln x}{x^4}$$

We will apply the chain rule to find $\frac{d}{dx}\ln(\ln x)$. Let $y = \ln(\ln x)$

Put $u = \ln x$, so that $y(u) = \ln u, \qquad u(x) = \ln x, \qquad \frac{d}{dx}\ln(\ln x) = \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}\ln u\right]\left[\frac{d}{dx}\ln x\right] = \left(\frac{1}{u}\right)\left(\frac{1}{x}\right) = \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) = \frac{1}{x\ln x}$$

Set B

We will apply the chain rule to find $\frac{d}{dx}\ln(x^2 + 5)$. Let $y = \ln(x^2 + 5)$

Put $u = x^2 + 5$, so that

$$y(u) = \ln u\ , \qquad u(x) = x^2 + 5, \qquad \frac{d}{dx}\ln(x^2 + 5) = \frac{dy}{dx} = ?$$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}\ln u\right]\left[\frac{d}{dx}(x^2 + 5)\right] = \left(\frac{1}{u}\right)(2x + 0) = \left(\frac{1}{u}\right)(2x) = \frac{2x}{x^2 + 5}$$

5 We will apply the chain rule to find $\frac{d}{d\theta}\ln|\sin\theta|$. Let $y = \ln|\sin\theta|$

Put $u = \sin\theta$, so that $y(u) = \ln u, \qquad u(\theta) = \sin\theta, \qquad \frac{d}{dt}\ln|\sin\theta| = \frac{dy}{d\theta} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left(\frac{d}{du}\ln u\right)\left(\frac{d}{d\theta}\sin\theta\right) = \left(\frac{1}{u}\right)(\cos\theta) = \frac{\cos\theta}{u} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Explanation:

- Note that the logarithmic function is real only when its argument is positive. Since $\sin\theta$ has both positive and negative values, the absolute value $|\sin\theta|$ ensures positive argument of $\ln|\sin\theta|$.

6 We will apply the chain rule to find $\frac{d}{dx}\log_7(2x + 5)$. Let $y = \log_7(2x + 5)$

Put $u = 2x + 5$, so that

$y(u) = \log_7 u, \quad u(x) = 2x + 5, \quad \frac{d}{dx}\log_7(2x + 5) = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}\log_7 u\right]\left[\frac{d}{dx}(2x + 5)\right] = \left[\frac{d}{du}\log_7 u\right](2) = 2\frac{d}{du}\log_7 u$$

Now apply the change of base formula, which gives $\log_7 u = \frac{\ln u}{\ln 7}$. Therefore,

$$\frac{d}{dx}(\log_7 u) = \frac{d}{dx}\left(\frac{\ln u}{\ln 7}\right) = \left(\frac{1}{\ln 7}\right)\left[\frac{d}{dx}(\ln u)\right] = \left(\frac{1}{\ln 7}\right)\left(\frac{1}{u}\right) = \frac{1}{u\ln 7}$$

Substitute the above value to get

$$\frac{dy}{dx} = 2\frac{d}{du}\log_7 u = 2\left(\frac{1}{u\ln 7}\right) = \frac{2}{(2x + 5)\ln 7}$$

Set C

7 We will apply the chain rule to find $\frac{d}{dx}5^{x^2}$. Let $y = 5^{x^2}$

Put $u = x^2$, so that $y(u) = 5^u, \qquad u(x) = x^2, \qquad \frac{d}{dx}5^{x^2} = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}5^u\right]\left[\frac{d}{dx}(x^2)\right] = \left[\frac{d}{du}5^u\right][2x]$$

To find $\frac{d}{du}5^u$, recall the formula $\frac{d}{du}b^u = b^u \ln b$ and note that in this case $b = 5$.

This gives $\frac{d}{du}5^u = 5^u \ln 5$. Substituting this value, we have

$$\frac{dy}{dx} = (5^u \ln 5)(2x) = \left(5^{x^2} \ln 5\right)(2x) = 2x\, 5^{x^2} \ln 5$$

[redacted]We will apply the chain rule to find $\frac{d}{dx}\ln\sqrt{x+3}$. Let $y = \ln\sqrt{x+3}$

Put $u = \sqrt{x+3}$, so that $y(u) = \ln u, \quad u(x) = \sqrt{x+3}, \quad \frac{d}{dx}\ln\sqrt{x+3} = \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}\ln u\right]\left[\frac{d}{dx}\sqrt{x+3}\right] = \left[\frac{1}{u}\right]\left[\frac{d}{dx}\sqrt{x+3}\right] = \left(\frac{1}{\sqrt{x+3}}\right)\left[\frac{d}{dx}\sqrt{x+3}\right]$$

We need to apply the chain rule once again to find $\frac{d}{dx}\sqrt{x+3}$. For this, let $z = \sqrt{x+3}$.

Put $v = x + 3$, so that $z(v) = \sqrt{v}, \quad v(x) = x + 3, \quad \frac{d}{dx}\sqrt{x+3} = \frac{dz}{dx} = ?$

Applying the chain rule, we have $\frac{dz}{dx} = \frac{dz}{dv}\frac{dv}{dx}$

$$= \frac{d}{dv}(\sqrt{v})\frac{dv}{dx} = \frac{d}{dv}(v^{1/2})\frac{d}{dx}(x+3) = \left(\frac{1}{2}v^{-1/2}\right)(1) = \frac{1}{2\sqrt{v}} = \frac{1}{2\sqrt{x+3}}$$

Substituting the above value of $\frac{d}{dx}\sqrt{x+3}\left[=\frac{dz}{dx}\right]$ in the expression for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{x+3}}\right)\left[\frac{1}{2\sqrt{x+3}}\right] = \frac{1}{2(\sqrt{x+3})^2} = \frac{1}{2(x+3)}$$

Explanation:

$\sqrt{v} = v^{1/2}$. Since $x^{-n} = \frac{1}{x^n}$, we have $v^{-1/2} = \frac{1}{v^{1/2}}$.

9 Apply the chain rule to find $\frac{d}{dx}\ln(\tan x + \sec x)$.

Let $y = \ln(\tan x + \sec x)$

Put $u = \tan x + \sec x$, so that $y = \ln u, \quad u(x) = \tan x + \sec x, \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(\ln u)\right]\left[\frac{d}{dx}(\tan x + \sec x)\right] = \left[\frac{1}{u}\right]\left[\frac{d}{dx}\tan x + \frac{d}{dx}\sec x\right]$$

$$= \left(\frac{1}{\tan x + \sec x}\right)\left[\sec^2 x + \sec x \ \tan x\right]$$

$$= \frac{1}{(\tan x + \sec x)}\left[\sec x(\sec x + \tan x)\right] = \frac{\sec x(\sec x + \tan x)}{(\tan x + \sec x)} = \sec x$$

9 DERIVATIVES OF HYPERBOLIC FUNCTIONS

Hyperbolic functions are defined in terms of exponential functions as follows

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Some important results involving hyperbolic functions are:

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh(-x) = -\sinh x, \qquad \cosh(-x) = \cosh x$$

$$\sinh 0 = 0, \qquad \cosh 0 = 1, \qquad \tanh 0 = 0$$

We now give below the derivatives of basic hyperbolic functions

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \frac{d}{dx}\cosh x = \sinh x, \qquad \frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

Example 1: Find the following derivative with respect to x.

$$\frac{d}{dx}x^2 \sinh x$$

Solution: We will apply the product rule to find $\frac{d}{dx}x^2 \sinh x$. For this, let

$$f(x) = x^2, \quad g(x) = \sinh x, \quad \text{so that} \quad \frac{d}{dx}x^2 \sinh x = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = x^2\left[\frac{d}{dx}\sinh x\right] + \sinh x\left[\frac{d}{dx}(x^2)\right]$$

$$= x^2(\cosh x) + \sinh x(2x) = x^2 \cosh x + 2x\sinh x$$
$$= x(x\cosh x + 2\sinh x)$$

Example 2: Find the following derivative with respect to x.

$$\frac{d}{dx} 5\tanh 3x$$

Solution: We will apply the chain rule to find $\frac{d}{dx} 5\tanh 3x$. Let $y = 5\tanh 3x$

Put $u = 3x$, so that

$$y(u) = 5\tanh u, \qquad u(x) = 3x, \qquad \frac{d}{dx} 5\tanh 3x = \frac{dy}{dx} = \ ?$$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(5\tanh u)\right]\left[\frac{d}{dx}(3x)\right] = \left[5\frac{d}{du}\tanh u\right]\left[\frac{d}{dx}(3x)\right]$$

$$= \left[5(\operatorname{sech}^2 u)\right](3) = (5\operatorname{sech}^2 u)(3) = (5\operatorname{sech}^2 3x)(3) = 15\operatorname{sech}^2 3x$$

Example 3: Find the following derivative with respect to t.

$$\frac{d}{dt}\sqrt{\cosh t}$$

Solution: We will apply the chain rule to find $\frac{d}{dt}\sqrt{\cosh t}$. Let $y = \sqrt{\cosh t}$

Put $u = \cosh t$, so that

$$y(u) = \sqrt{u}, \qquad u(t) = \cosh t, \qquad \frac{d}{dt}\sqrt{\cosh t} = \frac{dy}{dt} = \ ?$$

Applying the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = \left[\frac{d}{du}(\sqrt{u})\right]\left[\frac{d}{dt}\cosh t\right] = \left[\frac{d}{du}(u^{1/2})\right]\left[\frac{d}{dt}\cosh t\right]$$

$$= \left(\frac{1}{2}u^{-1/2}\right)(\sinh t) = \left(\frac{1}{2u^{1/2}}\right)(\sinh t)$$

$$= \frac{\sinh t}{2u^{1/2}} = \frac{\sinh t}{2\sqrt{\cosh t}}$$

Chapter 9 Exercises

Set A

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d}{dx}5\sinh\sqrt{x} =$$

2

$$\frac{d}{dx}\sinh(1 - \cosh x) =$$

3

$$\frac{d}{dx}\sinh^2 x\cosh x =$$

Set B

4

$$\frac{d}{dx}\frac{1}{(2x + 5\sinh x)^3} =$$

5

$$\frac{d}{dx}\tanh^2(x^2 - 4) =$$

6

$$\frac{d}{dx}\cosh[\sinh(x)] =$$

Chapter 9 Solutions

Set A

[redacted]We will apply the chain rule to find $\frac{d}{dx}5\sinh\sqrt{x}$. Let $y = 5\sinh\sqrt{x}$

Put $u = \sqrt{x}$, so that $y(u) = 5\sinh u, \qquad u(x) = \sqrt{x}, \qquad \frac{d}{dx}5\sinh\sqrt{x} = \frac{dy}{dx} = ?$

Applying the chain rule, we have $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(5\sinh u)\right]\left[\frac{d}{dx}(x^{1/2})\right]$

$$= [5(\cosh u)]\left(\frac{1}{2}x^{-1/2}\right) = (5\cosh u)\left(\frac{1}{2x^{1/2}}\right) = \frac{5\cosh\sqrt{x}}{2\sqrt{x}} = \frac{(5\cosh\sqrt{x})}{2\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \frac{5\sqrt{x}\cosh\sqrt{x}}{2x}$$

Explanation: $\sqrt{x} = x^{1/2}$ Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-1/2} = \frac{1}{x^{1/2}}$, $\sqrt{x}\sqrt{x} = \sqrt{x^2} = x$

- Both $\frac{5\cosh^{-1}\sqrt{x}}{2\sqrt{x}}$ and $\frac{5\sqrt{x}\cosh^{-1}\sqrt{x}}{2x}$ are correct answers of which $\frac{5\sqrt{x}\cosh^{-1}\sqrt{x}}{2x}$ has rational denominator.

[redacted]We will apply the chain rule to find $\frac{d}{dx}\sinh(1 - \cosh x)$. Let $y = \sinh(1 - \cosh x)$

Put $u = 1 - \cosh x$, so that $y(u) = \sinh u, \quad u(x) = 1 - \cosh x, \quad \frac{dy}{dx} = ?$

Apply the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(\sinh u)\right]\left[\frac{d}{dx}(1 - \cosh x)\right]$

$$= (\cosh u)[-\sinh x] = \cosh(1 - \cosh x)(-\sinh x) = -\sinh x\cosh(1 - \cosh x)$$

Explanation:

- Note that $\sinh(1 - \cosh x)$ does not imply multiplication of two functions. Instead, $(1 - \cosh x)$ is the argument of hyperbolic sine function.
- Note also the difference in sign between derivatives of hyperbolic cosine and normal cosine: $\frac{d}{dx}\cosh x = \sinh x$ whereas $\frac{d}{dx}\cos x = -\sin x$.

[redacted]We will apply the product rule to find $\frac{d}{dx}\sinh^2 x\cosh x$. For this, let

$$f(x) = \sinh^2 x, \quad g(x) = \cosh x, \quad \text{so that } \frac{d}{dx}\sinh^2 x\cosh x = \frac{d}{dx}(fg) = ?$$

Apply the product rule: $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$= \sinh^2 x\left(\frac{d}{dx}\cosh x\right) + \cosh x\left(\frac{d}{dx}\sinh^2 x\right) = \sinh^2 x\sinh x + \cosh x\left(\frac{d}{dx}\sinh^2 x\right)$$

To find $\frac{d}{dx}\sinh^2 x$, we will apply the chain rule. For this, let $z = \sinh^2 x$

Put $v = \sinh x$, so that $z(v) = v^2, \quad v(x) = \sinh x, \frac{d}{dx}\sinh^2 x = \frac{dz}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dz}{dx} = \frac{dz}{dv}\frac{dv}{dx} = \left[\frac{d}{du}(v^2)\right]\left[\frac{d}{dx}(\sinh x)\right] = (2v)\ (\cosh x) = 2\sinh x\cosh x$$

Substitute the above value of $\frac{d}{dx}\sinh^2 x\ \left[= \frac{dz}{dx}\right]$ in the expression for $\frac{d}{dx}(fg)$

$$\frac{d}{dx}(fg) = (\sinh^2 x)(\sinh x) + (\cosh x)[2\sinh x\cosh x]$$

$$= \sinh^3 x + 2\sinh x\cosh^2 x$$

Explanation:

- Remember the difference in sign between derivatives of hyperbolic cosine and normal cosine: $\frac{d}{dx}\cosh x = \sinh x$ whereas $\frac{d}{dx}\cos x = \ -\sin x$.

Set B

We will apply the chain rule to find $\frac{d}{dx}\frac{1}{(2x + 5\sinh x)^3}$. Let $y = \ \frac{1}{(2x + 5\sinh x)^3}$

Put $u = 2x + 5\sinh x$ so that $y(u) = \frac{1}{u^3}, \ u(x) = 2x + 5\sinh x, \ \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}\left(\frac{1}{u^3}\right)\right]\left[\frac{d}{dx}(2x + 5\sinh x)\right] = \left[\frac{d}{du}(u^{-3})\right]\left[\frac{d}{dx}(2x + 5\sinh x)\right]$$

$$= (-3u^{-4})[2 + 5\cosh x] = \left(\frac{-3}{u^4}\right)(2 + 5\cosh x)$$

$$= \frac{-3(2 + 5\cosh x)}{(2x + 5\sinh x)^4}$$

We will apply the chain rule to find $\frac{d}{dx}\tanh^2(x^2 - 4)$.

Let $y = \tanh^2(x^2 - 4) = [\tanh(x^2 - 4)]^2$

Put $u = \tanh(x^2 - 4)$, so that $y(u) = u^2, \quad u(x) = \tanh(x^2 - 4), \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^2)\right]\left[\frac{d}{dx}\tanh\left(x^2 - 4\right)\right] = (2u)\left[\frac{d}{dx}\tanh\left(x^2 - 4\right)\right]$$

$$= 2\tanh\left(x^2 - 4\right)\left[\frac{d}{dx}\tanh\left(x^2 - 4\right)\right]$$

Apply the chain rule once again to find the derivative of $\frac{d}{dx}\tanh\left(x^2 - 4\right)$. For this, let $z = \tanh\left(x^2 - 4\right)$

Put $v = x^2 - 4$, so that $z(v) = \tanh v,\ \ v(x) = x^2 - 4,\ \frac{d}{dx}\tanh\left(x^2 - 4\right) = \frac{dz}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dz}{dx} = \frac{dz}{dv}\frac{dv}{dx} = \left[\frac{d}{dv}(\tanh v)\right]\left[\frac{d}{dx}(x^2 - 4)\right] = \left(\mathrm{sech}^2 v\right)\left[\frac{d}{dx}(x^2 - 4)\right]$$

$$= \mathrm{sech}^2\left(x^2 - 4\right)(2x - 0) = 2x\,\mathrm{sech}^2\left(x^2 - 4\right)$$

Substituting the above value, we get

$$\frac{dy}{dx} = 2\tanh\left(x^2 - 4\right)\left[2x\,\mathrm{sech}^2\left(x^2 - 4\right)\right] = 4x\tanh\left(x^2 - 4\right)\mathrm{sech}^2\left(x^2 - 4\right)$$

[redacted]We will apply the chain rule to find $\frac{d}{dx}\cosh\left[\sinh(x)\right]$. Let $y = \ \cosh\left[\sinh(x)\right]$

Put $u = \sinh x$, so that $y(u) = \cosh u,\ u(x) = \sinh x,\ \frac{d}{dx}\cosh\left[\sinh(x)\right] = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(\cosh u)\right]\left[\frac{d}{dx}(\sinh x)\right] = \left[\sinh u\right]\ (\cosh x)$$

$$= \left[\sinh(\sinh x)\right]\ (\cosh x) = \cosh x \sinh\left[\sinh(x)\right]$$

Explanation:

- Note that $\cosh\left[\sinh(x)\right]$ does not imply multiplication of two functions. Instead, $\sinh(x)$ is the argument of hyperbolic cosine function.
- Note the difference in sign between derivatives of hyperbolic cosine and normal cosine: $\frac{d}{du}\cosh u = \sinh u$ whereas $\frac{d}{dx}\cos u = \ - \sin u$.

10 DERIVATIVES OF IMPLICIT FUNCTIONS

In the previous chapters, we have learnt how to find the derivatives of functions of the form $y = f(x)$ like $y = x^2 + 3$. Writing this function as an equation we have

$$y - x^2 - 3 = 0$$

In this case, we can solve for y and express the relationship as $y = x^2 + 3$. In such cases, when it is possible to solve for y and express the relationship between x and y the form $y = f(x)$, we say that y is given as an explicit function of x.

However, it is not always possible or convenient to express the functions in this form. For example, consider the following relationship

$$y^5 - \sin x + y = 0.$$

In this case, there is no easy way to express y in terms of x. In such cases, when a function is defined by means of a relationship that it does not seem possible to express y as a function of x in the form $y = f(x)$, we say that y is an implicit function of x.

The process of finding the derivative of an implicit function is known as implicit differentiation. We begin by finding the derivative of each term on both sides of the defining equation with respect to x. While doing so in the above example, we need to find the derivative of y^5. We will make an intelligent use of the chain rule to calculate $\frac{d}{dx}(y^5)$. First, notice that y^5 is a function of y. Also, y itself is a function of the variable x, (albeit an implicit function where we cannot isolate y in terms of x). As a result, y^5 becomes a function of another function- a composite function.

Now, for a composite function $z(y(x))$, the chain rule says

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

Substituting $z = y^5$

$$\frac{d}{dx}(y^5) = \frac{d}{dy}(y^5)\frac{dy}{dx} = 5y^4\frac{dy}{dx}$$

To find $\frac{dy}{dx}$ using implicit differentiation, follow these steps:

1. Find the derivative of each term on both sides of the defining equation with respect to x.
2. Transfer all terms containing $\frac{dy}{dx}$ to the left-hand side and the remaining terms to the other side.
3. Factor out $\frac{dy}{dx}$ and divide.

Now, study carefully the following examples and then attempt the exercises to gain confidence in implicit differentiation.

Example 1: Find $\frac{dy}{dx}$ if

$$y^5 - \sin x + y = 0$$

Solution: We will use implicit differentiation to find $\frac{dy}{dx}$.

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(y^5) - \frac{d}{dx}(\sin x) + \frac{d}{dx}(y) = \frac{d}{dx}(0) \rightarrow \frac{d}{dx}(y^5) - \cos x + \frac{dy}{dx} = 0$$

Let $z = y^5$ so that z is a function of y and $\frac{d}{dx}(y^5) = \frac{dz}{dx} = ?$

Using implicit differentiation, we get

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \left[\frac{d}{dy}(y^5)\right]\left[\frac{dy}{dx}\right] = (5y^4)\frac{dy}{dx} = 5y^4\frac{dy}{dx}$$

Substituting the above value, we get

$$5y^4\frac{dy}{dx} - \cos x + \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx}(5y^4 + 1) = \cos x \rightarrow \frac{dy}{dx} = \frac{\cos x}{(5y^4 + 1)}$$

Note that the derivative of implicit function contains terms containing both x and y unlike in case of derivatives of explicit functions, which are defined in terms of x alone.

Example 2: Find $\frac{dy}{dx}$ if

$$y - x^4 + y^3 = 0$$

Solution: We will use implicit differentiation to find $\frac{dy}{dx}$.

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(y) - \frac{d}{dx}(x^4) + \frac{d}{dx}(y^3) = \frac{d}{dx}(0)$$

$$\frac{dy}{dx} - (4x^3) + \frac{d}{dx}(y^3) = 0 \rightarrow \frac{dy}{dx} - 4x^3 + \frac{d}{dx}(y^3) = 0$$

Let $z = y^3$ so that z is a function of y and $\frac{d}{dx}(y^3) = \frac{dz}{dx} = ?$ Use implicit differentiation:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \left[\frac{d}{dy}(y^3)\right]\left[\frac{dy}{dx}\right] = (3y^2)\frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

Substituting the above value, we get

$$\frac{dy}{dx} - 4x^3 + 3y^2\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx}(1 + 3y^2) - 4x^3 = 0$$

$$\frac{dy}{dx}(1 + 3y^2) = 4x^3 \rightarrow \frac{dy}{dx} = \frac{4x^3}{1 + 3y^2}$$

Example 3: Find $\frac{dy}{dx}$ if

$$3x^2 + 2xy + 4y^4 = 0$$

Solution: We have

$$3x^2 + 2xy + 4y^4 = 0$$

We will use implicit differentiation to find $\frac{dy}{dx}$.

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(3x^2) + \frac{d}{dx}(2xy) + \frac{d}{dx}(4y^4) = \frac{d}{dx}(0) \rightarrow 6x + 2\frac{d}{dx}(xy) + 4\frac{d}{dx}(y^4) = 0$$

Apply the product rule to find $\frac{d}{dx}(xy)$. Let $f(x) = x,\ \ g(x) = y,\ \ \frac{d}{dx}(xy) = \frac{d}{dx}(fg) = ?$

Applying the product rule, we have $\frac{d}{dx}(xy) = \frac{d}{dx}(fg)$

$$= f\frac{dg}{dx} + g\frac{df}{dx} = x\left[\frac{d}{dx}(y)\right] + y\left[\frac{d}{dx}(x)\right] = x\frac{dy}{dx} + y(1) = x\frac{dy}{dx} + y$$

Let $z = y^4$ so that z is a function of y and $\frac{d}{dx}(y^4) = \frac{dz}{dx} = \ ?$ Use implicit differentiation:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \left[\frac{d}{dy}(y^4)\right]\left[\frac{dy}{dx}\right] = (4y^3)\frac{dy}{dx} = 4y^3\frac{dy}{dx}$$

Substituting the above values, we get

$$6x + 2\left(x\frac{dy}{dx} + y\right) + 4\left(4y^3\frac{dy}{dx}\right) = 0$$

$$6x + 2x\frac{dy}{dx} + 2y + 16y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x + 16y^3) + 6x + 2y = 0$$

$$\frac{dy}{dx}(2x + 16y^3) = -6x - 2y$$

$$\frac{dy}{dx} = \frac{-6x - 2y}{(2x + 16y^3)} = \frac{-2(3x + y)}{2(x + 8y^3)} = -\frac{3x + y}{x + 8y^3}$$

Note that $\frac{dy}{dx}$ contains terms containing both x and y.

Chapter 10 Exercises

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

- Find $\frac{dy}{dx}$ if $\cos x + \tan y = y$

- Find $\frac{dy}{dx}$ if $y^3 + x^3 - 3y = 27$

- Find $\frac{dy}{dx}$ if $x + y^2 - \cos y = 1$

Chapter 10 Solutions

[redacted]We have $\cos x + \tan y = y$. We will use implicit differentiation to find $\frac{dy}{dx}$.

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\cos x + \frac{d}{dx}\tan y = y$$

$$-\sin x + \frac{d}{dx}\tan y = \frac{dy}{dx} \quad \rightarrow \quad -\sin x + \frac{d}{dx}\tan y - \frac{dy}{dx} = 0$$

Let $z = \tan y$ so that $\frac{d}{dx}\tan y = \frac{dz}{dx} = \ ?$ Using implicit differentiation, we get

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \left[\frac{d}{dy}\tan y\right]\left[\frac{dy}{dx}\right] = \left(\sec^2 y\right)\frac{dy}{dx} = \sec^2 y\frac{dy}{dx}$$

Substituting the value of $\frac{dz}{dx}\left(= \frac{d}{dx}\tan y\right)$, we get

$$-\sin x + \sec^2 y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(\sec^2 y - 1\right) = \sin x \quad \rightarrow \quad \frac{dy}{dx} = \frac{\sin x}{\left(\sec^2 y - 1\right)} = \frac{\sin x}{\tan^2 y}$$

Explanation:

- $\sec^2 y - 1 = \tan^2 y$ because $\tan^2 y + 1 = \sec^2 y$

[redacted]We have $y^3 + x^3 - 3y = 27$. We will use implicit differentiation to find $\frac{dy}{dx}$.

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}y^3 + \frac{d}{dx}x^3 - \frac{d}{dx}(3y) = \frac{d}{dx}(27) \quad \rightarrow \quad \frac{d}{dx}y^3 + 3x^2 - 3\frac{dy}{dx} = 0$$

Let $z = y^3$ so that $\frac{d}{dx}y^3 = \frac{dz}{dx} = \ ?$ Using implicit differentiation, we get

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \left[\frac{d}{dy}y^3\right]\left[\frac{dy}{dx}\right] = \left(3y^2\right)\frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

Substituting the above values of $\frac{d}{dx}y^3$, we get

$$3y^2\frac{dy}{dx} + 3x^2 - 3\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx}\left(3y^2 - 3\right) + 3x^2 = 0 \rightarrow \frac{dy}{dx}\left(3y^2 - 3\right) = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2 - 3} = \frac{-3x^2}{3(y^2 - 1)} = \frac{-x^2}{y^2 - 1}$$

[redacted]We have $x + y^2 - \cos y = 1$. We will use implicit differentiation to find $\frac{dy}{dx}$.
Differentiating both sides with respect to x, we get

$$\frac{d}{dx}x + \frac{d}{dx}y^2 - \frac{d}{dx}\cos y = \frac{d}{dx}(1)$$

$$1 + \frac{d}{dx}y^2 - \frac{d}{dx}\cos y = 0$$

Let $z = y^2$, $w = \cos y$ so that $\frac{d}{dx}y^2 = \frac{dz}{dx} = ?$ and $\frac{d}{dx}\cos y = \frac{dw}{dx} = ?$
Using implicit differentiation, we get

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \left[\frac{d}{dx}y^2\right]\left[\frac{dy}{dx}\right] = (2y)\frac{dy}{dx} = 2y\frac{dy}{dx} \text{ and}$$

$$\frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = \left[\frac{d}{dx}\cos y\right]\left[\frac{dy}{dx}\right] = (-\sin y)\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

Substituting the above values of $\frac{d}{dx}y^2$ and $\frac{d}{dx}\cos y$, we get

$$1 + 2y\frac{dy}{dx} - \left(-\sin y\frac{dy}{dx}\right) = 0$$

$$1 + 2y\frac{dy}{dx} + \sin y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y + \sin y) = -1$$

$$\frac{dy}{dx} = -\frac{1}{2y + \sin y}$$

11 SECOND ORDER DERIVATIVES

In previous chapters, we have learnt how to find the derivative $\frac{dy}{dx}$ of $y = f(x)$ with respect to x. We call $\frac{dy}{dx}$ as the first order derivative of y with respect to x. In general, the first order derivative is itself a function of x and, therefore, we can again find its derivative with respect to x. Such a derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ of $\frac{dy}{dx}$ is termed as the second order derivative of y with respect to x and is denoted as $\frac{d^2y}{dx^2}$.

Study carefully the following examples to fix up the idea and then attempt the exercises given at the end of the chapter.

Example 1: Find the following derivative with respect to x.

$$\frac{d^2}{dx^2}\left(4x^3\right)$$

Solution: Let $y = 4x^3$

$$\frac{dy}{dx} = \frac{d}{dx}\left(4x^3\right) = 12x^2$$

Now, taking the second derivative, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(12x^2\right) = 24x$$

Example 2: Find the following derivative with respect to x.

$$\frac{d^2}{dx^2}\left(5x^4 + 6x^3 + 4x^2 - 8x\right)$$

Solution: Let $y = 5x^4 + 6x^3 + 4x^2 - 8x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(5x^4 + 6x^3 + 4x^2 - 8x\right) = 20x^3 + 18x^2 + 8x - 8$$

Now, taking the second derivative, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(20x^3 + 18x^2 + 8x - 8\right) = 60x^2 + 36x + 8 - 0$$

$$= 60x^2 + 36x + 8$$

Example 3: Find the following derivative with respect to x.

$$\frac{d^2}{dx^2}(\cot x)$$

Solution: Let $y = \cot x$

$$\frac{dy}{dx} = \frac{d}{dx}(\cot x) = -\csc^2 x$$

Now, taking the second derivative, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\csc^2 x)$$

We will apply the chain rule to find $\frac{d}{dx}(-\csc^2 x)$. Let $z = -\csc^2 x$. Put $u = \csc x$, so that

$$z(u) = -u^2, \qquad u(x) = \csc x, \qquad \frac{d}{dx}(-\csc^2 x) = \frac{dz}{dx} = ?$$

Applying the chain rule, we have

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = \frac{dz}{du}\frac{du}{dx} = \frac{d}{du}(-u^2)\frac{du}{dx} = (-2u)\left[\frac{d}{dx}(\csc x)\right]$$

$$= (-2\csc x)[-\csc x \cot x] = 2\csc^2 x \cot x$$

Example 4: Find the following derivative with respect to x.

$$\frac{d^2}{dx^2}(3\tan^{-1} x)$$

Solution: Let $y = 3\tan^{-1} x$

$$\frac{dy}{dx} = \frac{d}{dx}(3\tan^{-1} x) = \frac{3}{1 + x^2} = 3(1 + x^2)^{-1}.$$

Take the second derivative: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[3(1 + x^2)^{-1}\right]$

To find the derivative of $\left[3(1 + x^2)^{-1}\right]$ we apply the chain rule. Let $z = 3(1 + x^2)^{-1}$.

Now, put $u = 1 + x^2$, so that $z(u) = 3u^{-1}, \; u(x) = 1 + x^2, \; \frac{dz}{dx} = ?$

Applying the chain rule, we have $\frac{d^2y}{dx^2} = \frac{dz}{dx} = \frac{dz}{du}\frac{du}{dx} = \frac{d}{du}\left(3u^{-1}\right)\frac{d}{dx}\left(1 + x^2\right)$

$$=-3u^{-2}(0 + 2x) =-\frac{3}{u^2}(2x) =-\frac{3}{\left(1 + x^2\right)^2}(2x) = -\frac{6x}{\left(1 + 2x^2 + x^4\right)}$$

Chapter 11 Exercises

Set A

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\frac{d^2}{dx^2}\left(x^6 + 6x^4 + 5x^2 + 7x\right) =$$

2

$$\frac{d^2}{dx^2}\left(\frac{1}{x^2}\right) =$$

3

$$\frac{d^2}{d\theta^2}4\sin 3\theta =$$

Set B

4

$$\frac{d^2}{dx^2}\frac{1}{x\sqrt{x}} =$$

5

$$\frac{d^2}{d\theta^2}\cos\left(\theta^3\right) =$$

6

$$\frac{d^2}{dx^2}x^2e^x =$$

Set C

7

$$\frac{d^2}{dx^2} x \ln x =$$

8

$$\frac{d^2}{dx^2} \cot^{-1} x =$$

$$\frac{d^2}{dx^2} 4e^{3x^2} =$$

Set D

10

$$\frac{d^2}{dx^2} (x \cos x) =$$

11

$$\frac{d^2}{dx^2} (\ln x + \sqrt{x}) =$$

12

$$\frac{d^2}{dx^2} \sin(x^2 + 3) =$$

Chapter 11 Solutions

1 Set A

Let $y = x^6 + 6x^4 + 5x^2 + 7x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^6 + 6x^4 + 5x^2 + 7x) = 6x^5 + 24x^3 + 10x + 7$$

Now, taking the second derivative, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(6x^5 + 24x^3 + 10x + 7) = 30x^4 + 72x^2 + 10 + 0$$

$$= 30x^4 + 72x^2 + 10$$

2 Let $y = \frac{1}{x^2} = x^{-2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = (-2)x^{-2-1} = -2x^{-3}$$

Now, taking the second derivative, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-2x^{-3}) = (-2)(-3)x^{-3-1} = 6x^{-4} = \frac{6}{x^4}$$

Explanation:

- $(-2)(-3) = 6.$
- Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-4} = \frac{1}{x^4}$. Both $6x^{-4}$ and $\frac{6}{x^4}$ are correct answers.

3 Let $y = 4\sin 3\theta$

We will apply the chain rule to find $\frac{d}{d\theta}(4\sin 3\theta)$.

Put $u = 3\theta$, so that $y(u) = 4\sin u, \qquad u(\theta) = 3\theta, \qquad \frac{d}{d\theta}(4\sin 3\theta) = \frac{dy}{dx} = \ ?$

Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(4\sin u)\right]\left[\frac{d}{d\theta}(3\theta)\right] = (4\cos u)(3) = 12\cos u = 12\cos 3\theta$$

Now, taking the second derivative, we have

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = \frac{d}{d\theta}(12\cos 3\theta)$$

To find $\frac{d}{d\theta}(12\cos 3\theta)$, we apply the chain rule once more. Let $z = 12\cos 3\theta$

Put $v = 3\theta$, so that

$$z(v) = 12\cos v, \qquad v(\theta) = 3\theta, \qquad \frac{d}{d\theta}(12\cos 3\theta) = \frac{dz}{d\theta} = \ ?$$

Applying the chain rule, we have

$$\frac{d^2y}{d\theta^2} = \frac{dz}{d\theta} = \frac{dz}{dv}\frac{dv}{d\theta} = \left[\frac{d}{dv}(12\cos v)\right]\left[\frac{d}{d\theta}(3\theta)\right] = (-12\sin v)(3)$$

$$= (-12\sin 3\theta)(3) = -36\sin 3\theta$$

Set B

[redacted]

Let $y = \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}, \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^{3/2}}\right) = \frac{d}{dx}\left(x^{-3/2}\right) = -\frac{3}{2}x^{-5/2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{3}{2}x^{-5/2}\right) = -\frac{3}{2}\frac{d}{dx}\left(x^{-5/2}\right) = \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2}$$

$$= \frac{15}{4}\frac{1}{x^{7/2}} = \frac{15}{4x^3\sqrt{x}} = \frac{15}{4x^3\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \frac{15\sqrt{x}}{4x^4}$$

Explanation:

- Since $x^m x^n = x^{m+n}$, we have $x\sqrt{x} = x^1 x^{1/2} = x^{1+1/2} = x^{3/2}$.
- $-\frac{3}{2} - 1 = -\frac{3}{2} - \frac{2}{2} = \frac{-3-2}{2} = -\frac{5}{2}$ and $-\frac{5}{2} - 1 = -\frac{5}{2} - \frac{2}{2} = \frac{-5-2}{2} = -\frac{7}{2}$
- Since $x^{-n} = \frac{1}{x^n}$, we have $x^{-7/2} = \frac{1}{x^{7/2}}$.
- $x^{7/2} = x^{3+1/2} = x^3 x^{1/2} = x^3\sqrt{x}$, since $x^{m+n} = x^m x^n$.
- $\frac{15}{4}\frac{1}{x^{7/2}}, \frac{15}{4x^3\sqrt{x}}$ and $\frac{15\sqrt{x}}{4x^4}$ are all correct answers. However, only $\frac{15\sqrt{x}}{4x^4}$ has rational denominator.

5

Let $y = \cos(\theta^3)$. We will apply the chain rule to find $\frac{d}{d\theta}\cos(\theta^3)$.

Put $u = \theta^3$, so that $y(u) = \cos u, \qquad u(\theta) = \theta^3, \qquad \frac{d}{d\theta}\cos(\theta^3) = \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(\cos u)\right]\left[\frac{d}{d\theta}(\theta^3)\right] = (-\sin u)(3\theta^2) = -3\theta^2\sin\theta^3$$

Now, take the second derivative:

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = \frac{d}{d\theta}\left(-3\theta^2\sin\theta^3\right)$$

To find $\frac{d}{d\theta}\left(-3\theta^2 \sin\theta^3\right),$ we need to apply the product rule. For this, let

$$f(\theta) = -3\theta^2,\ \ g(\theta) = \sin\theta^3 \text{ so that } \frac{d}{d\theta}\left(-3\theta^2 \sin\theta^3\right) = \frac{d}{d\theta}(fg) = ?$$

Applying the product rule, we have

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(fg) = f\frac{dg}{d\theta} + g\frac{df}{d\theta} = \left(-3\theta^2\right)\left[\frac{d}{d\theta}\left(\sin\theta^3\right)\right] + \left(\sin\theta^3\right)\left[\frac{d}{d\theta}\left(-3\theta^2\right)\right]$$

$$= \left(-3\theta^2\right)\left[\frac{d}{d\theta}\left(\sin\theta^3\right)\right] + \left(\sin\theta^3\right)[-6\theta]$$

To find $\frac{d}{d\theta}\left(\sin\theta^3\right),$ we need to reapply the chain rule. Let $z = \sin\theta^3$

We have $u = \theta^3$, so that $z(u) = \sin u,\quad u(\theta) = \theta^3,\quad \frac{d}{d\theta}\sin\theta^3 = \frac{dz}{d\theta} = ?$

Applying the chain rule, we have

$$\frac{dz}{d\theta} = \frac{dz\,du}{du\,d\theta} = \frac{d}{dv}(\sin u)\frac{d}{d\theta}\left(\theta^3\right) = (\cos u)\left(3\theta^2\right) = \left(\cos\theta^3\right)\left(3\theta^2\right) = 3\theta^2\cos\theta^3$$

Substitute the above value of $\frac{dz}{d\theta}\left(= \frac{d}{d\theta}\left(\sin\theta^3\right)\right)$ in the expression for $\frac{d}{d\theta}(fg)$.

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(fg) = \left(-3\theta^2\right)\left[3\theta^2\cos\theta^3\right] + \left(\sin\theta^3\right)[-6\theta]$$

$$= -9\theta^4\cos\theta^3 - 6\theta\sin\theta^3 = -3\theta\left(3\theta^3\cos\theta^3 + 2\sin\theta^3\right)$$

Explanation:

- Note that θ^3 is the argument of cosine function.
- Both $-9\theta^4\cos\theta^3 - 6\theta\sin\theta^3$ and $-3\theta\left(3\theta^3\cos\theta^3 + 2\sin\theta^3\right)$ are correct answers.

6 Let $y = x^2e^x$

We will apply the product rule to find $\frac{d}{dx}\left(x^2e^x\right)$. For this,

$$f(x) = x^2,\ \ g(x) = e^x \text{ so that } \frac{d}{dx}\left(x^2e^x\right) = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have $\frac{dy}{dx} = \frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$= x^2\left[\frac{d}{dx}e^x\right] + e^x\left[\frac{d}{dx}\left(x^2\right)\right] = x^2\left(e^x\right) + e^x(2x) = x^2e^x + 2xe^x = \left(x^2 + 2x\right)e^x$$

Now, take the second derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[(x^2 + 2x)e^x\right]$$

To find $\frac{d}{dx}\left[(x^2 + 2x)e^x\right]$, we need to apply the product rule. For this, let $h(x) = x^2 + 2x$. We also have $g(x) = e^x$, so that $\frac{d}{dx}\left[(x^2 + 2x)e^x\right] = \frac{d}{dx}(hg) = ?$

Applying the product rule, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(hg) = h\frac{dg}{dx} + g\frac{dh}{dx} = (x^2 + 2x)\left[\frac{d}{dx}e^x\right] + e^x\left[\frac{d}{dx}(x^2 + 2x)\right]$$

$$= (x^2 + 2x)e^x + e^x(2x + 2) = e^x(x^2 + 2x + 2x + 2) = e^x\left(x^2 + 4x + 2\right)$$

Explanation:

- $(x^2 + 2x)e^x = e^x(2x + 2)$, since multiplication is commutative.

Set C

7 Let $y = x\ln x$

We will apply the product rule to find the derivative of *y* with respect to *x*. For this, let

$f(x) = x,\ g(x) = \ln x$ so that $\frac{d}{dx}(x\ln x) = \frac{d}{dx}(fg) = ?$

Applying the product rule, we have $\frac{dy}{dx} = \frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$= x\left[\frac{d}{dx}\ln x\right] + \ln x\left[\frac{d}{dx}(x)\right] = x\left(\frac{1}{x}\right) + \ln x(1) = 1 + \ln x$$

Now, take the second derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(1 + \ln x) = \frac{d}{dx}(1) + \frac{d}{dx}(\ln x) = 0 + \frac{1}{x} = \frac{1}{x}$$

8 Let $y = \cot^{-1} x$.

$$\frac{dy}{dx} = \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2} = -(1 + x^2)^{-1}.$$

Take the second derivative: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(1 + x^2)^{-1}\right]$

To find $\frac{d}{dx}\left[-(1 + x^2)^{-1}\right]$, we apply the chain rule. Let $z = -(1 + x^2)^{-1}$.

Put $u = 1 + x^2$, so that $z(u) =- u^{-1}$, $u(x) = 1 + x^2$, $\frac{d}{dx}\left[-\left(1 + x^2\right)^{-1}\right] = \frac{dz}{dx} = ?$

Applying the chain rule, we have $\frac{d^2y}{dx^2} = \frac{dz}{dx} = \frac{dz}{du}\frac{du}{dx} = \frac{d}{du}\left(-u^{-1}\right)\frac{d}{dx}\left(1 + x^2\right)$

$$=-\left(-u^{-2}\right)(0 + 2x) = u^{-2}(2x) = \frac{1}{u^2}(2x) = \frac{2x}{(1 + x^2)^2} = \frac{2x}{\left(1 + 2x^2 + x^4\right)}$$

9

Let $y = 4e^{3x^2}$. We will apply the chain rule to find $\frac{dy}{dx}$.

Put $u = 3x^2$, so that $y(u) = 4e^u$, $u(x) = 3x^2$, $\frac{d}{dx}\left(4e^{3x^2}\right) = \frac{dy}{dx} = ?$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(4e^u)\right]\left[\frac{d}{dx}(3x^2)\right] = (4e^u)\,(6x) = \left(4e^{3x^2}\right)(6x) = 24xe^{3x^2}$$

Take the second derivative: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(24xe^{3x^2}\right)$

We will now apply the product rule to find $\frac{d}{dx}\left(24xe^{3x^2}\right)$. For this, let

$$f(x) = 24x,\ g(x) = e^{3x^2} \text{ so that } \frac{d}{dx}\left(24xe^{3x^2}\right) = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have $\frac{d^2y}{dx^2} = \frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$= (24x)\left[\frac{d}{dx}\left(e^{3x^2}\right)\right] + \left(e^{3x^2}\right)\left[\frac{d}{dx}(24x)\right] = (24x)\left[\frac{d}{dx}\left(e^{3x^2}\right)\right] + \left(e^{3x^2}\right)(24)$$

We have already obtained the result $\frac{dy}{dx} = \frac{d}{dx}\left(4e^{3x^2}\right) = 4\frac{d}{dx}\left(e^{3x^2}\right) = 24xe^{3x^2}$.

Therefore, $\frac{d}{dx}\left(e^{3x^2}\right) = \frac{24}{4}xe^{3x^2} = 6xe^{3x^2}$. S ubstituting the value of $\frac{d}{dx}e^{3x^2}$, we get

$$\frac{d^2y}{dx^2} = (24x)\left(6xe^{3x^2}\right) + e^{3x^2}(24) = 24\,e^{3x^2}\left(6x^2 + 1\right)$$

Explanation:

- Note that $e^{3x^2}(24) = 24e^{3x^2}$.
- Also note that we have used the constant multiple rule to obtain $\frac{d}{dx}\left(4e^{3x^2}\right) = 4\frac{d}{dx}\left(e^{3x^2}\right)$.

Set D

[redacted] Let $y = x\cos x$. Let $f(x) = x,\ \ g(x) = \cos x\ ,\ \text{ so that } \frac{d}{dx}(x\cos x) = \frac{d}{dx}(fg) = ?$

Applying the product rule, we have $\frac{dy}{dx} = \frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$= x\left[\frac{d}{dx}\cos x\right] + \cos x\left[\frac{d}{dx}x\right] = x(-\sin x) + \cos x(1) = -x\sin x + \cos x$$

Take the second derivative: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dx}(-x\sin x) + \frac{d}{dx}\cos x = \frac{d}{dx}(-x\sin x) + (-\sin x) = -\frac{d}{dx}(x\sin x) - \sin x$$

We will reapply the product rule to find $\frac{d}{dx}(x\sin x)$. We have $f(x) = x$.

Let $h(x) = \sin x\ ,\ \text{ so that } \frac{d}{dx}(x\sin x) = \frac{d}{dx}(fh) = ?$

Applying the product rule, we have $\frac{d}{dx}(fh) = f\frac{dh}{dx} + h\frac{df}{dx}$

$$= x\left[\frac{d}{dx}\sin x\right] + \sin x\left[\frac{d}{dx}x\right] = x(\cos x) + \sin x(1) = x\cos x + \sin x$$

Substituting the value of $\frac{d}{dx}(x\sin x)\left[= \frac{d}{dx}(fh)\right]$ in the expression for $\frac{d^2y}{dx^2}$, we have

$$\frac{d^2y}{dx^2} = -(x\cos x + \sin x) - \sin x = -x\cos x - \sin x - \sin x = -x\cos x - 2\sin x$$

11 Let $y = \ln x + \sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\ln x + \sqrt{x}) = \frac{d}{dx}(\ln x) + \frac{d}{dx}(x^{1/2}) = \frac{1}{x} + \frac{1}{2}x^{-1/2} = x^{-1} + \frac{1}{2}x^{-1/2}$$

Now, taking the second derivative, we have $\frac{d^2y}{dx^2}$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(x^{-1} + \frac{1}{2}x^{-1/2}\right)$$

$$= \frac{d}{dx}(x^{-1}) + \frac{d}{dx}\left(\frac{1}{2}x^{-1/2}\right) = -x^{-2} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-3/2} = -x^{-2} - \frac{x^{-3/2}}{4}$$

$$= -\frac{1}{x^2} - \frac{1}{4x^{3/2}} = -\frac{1}{x^2} - \frac{1}{4x\sqrt{x}} = -\frac{1}{x^2} - \frac{1}{4x\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = -\frac{4}{4x^2} - \frac{\sqrt{x}}{4x^2} = -\frac{4 + \sqrt{x}}{4x^2}$$

Explanation:

- $-\frac{1}{2}-1=-\frac{1}{2}-\frac{2}{2}=\frac{-1-2}{2}=-\frac{3}{2}$. Since $x^{m+n}=x^m x^n$, we have $x^{\frac{3}{2}}=x^{1+\frac{1}{2}}=x^1 x^{\frac{1}{2}}=x\sqrt{x}$. Also, $x\sqrt{x}\sqrt{x}=xx=x^2$.
- $-x^{-2}-\frac{x^{-3/2}}{4}, -\frac{1}{x^2}-\frac{1}{4x^{3/2}}$ and $-\frac{(4+\sqrt{x})}{4x^2}$ are all correct answers. However, only $-\frac{(4+\sqrt{x})}{4x^2}$ has rational denominator.

12 Let $y=\sin(x^2+3)$

We will apply the chain rule to find $\frac{dy}{dx}$.

Put $u=x^2+3$, so that

$y(u)=\sin u, \quad u(x)=x^2+3, \quad \frac{d}{dx}\sin(x^2+3)=\frac{dy}{dx}=\ ?$

Applying the chain rule, we have

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\left[\frac{d}{du}(\sin u)\right]\left[\frac{d}{dx}(x^2+3)\right]=(\cos u)\ (2x)=2x\cos(x^2+3)$$

Take the second derivative: $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left[2x\cos(x^2+3)\right]$

We will now apply the product rule to find $\frac{d}{dx}\left[2x\cos(x^2+3)\right]$. Let

$f(x)=2x, \quad g(x)=\cos(x^2+3)$, so that $\frac{d}{dx}\left[2x\cos(x^2+3)\right]=\frac{d}{dx}(fg)=\ ?$

Applying the product rule, we have $\frac{d^2y}{dx^2}=\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$

$$=(2x)\left[\frac{d}{dx}\cos(x^2+3)\right]+\cos(x^2+3)\left[\frac{d}{dx}(2x)\right]$$

$$=(2x)\left[\frac{d}{dx}\cos(x^2+3)\right]+\cos(x^2+3)(2)$$

To find $\frac{d}{dx}\cos(x^2+3)$, we need to apply the chain rule. Let $z=\cos(x^2+3)$

Put $v=x^2+3$, so that

$z(v)=\cos v, \quad v(x)=x^2+3, \quad \frac{d}{dx}\cos(x^2+3)=\frac{dz}{dx}=\ ?$

Applying the chain rule, we have

$$\frac{dz}{dx} = \frac{dzdv}{dvdx} = \frac{d}{dv}(\cos v)\,\frac{d}{dx}(x^2 + 3) = (-\sin v)(2x) = -2x\sin(x^2 + 3)$$

Finally, substituting the value of $\frac{d}{dx}\cos(x^2 + 3)\left(=\frac{dz}{dx}\right)$, we get

$$\frac{d^2y}{dx^2} = (2x)[-2x\sin(x^2 + 3)] + \cos(x^2 + 3)(2) = -4x^2\sin(x^2 + 3) + 2\cos(x^2 + 3)$$

Explanation:

- Since the derivative of a constant is zero, we have $\frac{d}{dx}(3) = 0$.

12 EXTREME VALUES-MAXIMA AND MINIMA

In order to determine the points of local maximum and local minimum of a function $f(x)$ and to find the value of the function $f(x)$ at these points, we follow the following steps:

1. Find the first derivative $\frac{df}{dx}$ of $f(x)$.
2. Put $\frac{df}{dx} = 0$. Geometrically $\frac{df}{dx} = 0$ implies that the slope of the tangent to the curve $y = f(x)$ at the point of local maximum or local minimum is zero or in other words, the tangent at such a point is parallel to the x-axis.
3. Solve the equation $\frac{df}{dx} = 0$ for x. Let $c_1, c_2, \ldots, c_n$ be the roots of the equation. Then $x = c_1, x = c_2, \ldots, x = c_n$ are the possible points where $f(x)$can attain a local maximum or local minimum.
4. Now, find the second derivative $\frac{d^2f}{dx^2}$ of $f(x)$.
5. Consider the first root $x = c_1$ found in step 3 above. Evaluate $\frac{d^2f}{dx^2}$ at $x = c_1$.

- If $\frac{d^2f}{dx^2} < 0$ at $x = c_1$, then $x = c_1$ is a points of local maximum.
- If $\frac{d^2f}{dx^2} > 0$ at $x = c_1$, then $x = c_1$ is a points of local minimum.
- If $\frac{d^2f}{dx^2} = 0$ at $x = c_1$, then follow the following step:

Determine the sign of $\frac{df}{dx}$ at a point just before $x = c_1$ and at a point just after $x = c_1$.

(a) If the sign of $\frac{df}{dx}$ changes sign from positive to negative from just before $x = c_1$ to just after $x = c_1$, then $x = c_1$ is a point of local maximum.

(b) If the sign of $\frac{df}{dx}$ changes sign from negative to positive from just before $x = c_1$ to just after $x = c_1$, then $x = c_1$ is a point of local maximum.

(c) If the sign of $\frac{df}{dx}$ does not change sign from just before $x = c_1$ to just after $x = c_1$, then $x = c_1$ is neither a point of local maximum nor a point of local minimum and is known as a point of inflection.

Test the remaining roots $x = c_2$, …, $x = c_n$ one by one by following the procedure in Step 5 to find all the points of local maximum and local minimum of a function $f(x)$.

6. Evaluate the value of $f(x)$ at each point of local maximum and local minimum obtained above. This gives the values of the function $f(x)$ at every such point.

If we wish to find the values of absolute maximum and minimum of the function $f(x)$ in the closed interval $[a, b]$, we follow the following steps:

(i) First, find the values of the function $f(x)$ at every point of local maximum and local minimum by following the steps 1 to 6 as given above.

(ii) Now, find the values $f(a)$ and $f(b)$ at the end points of the interval $[a, b]$.

(iii) Compare the values $f(a)$ and $f(b)$ with those at points of local maximum and local minimum in step (i). The largest among them is the absolute maximum and the smallest them is the absolute minimum of the function $f(x)$ in the closed interval $[a, b]$

Example 1: Find the absolute extrema of the following function in the interval $[0, 3]$

$$f(x) = x^3 - 12x - 45$$

Solution: We have $f(x) = x^3 - 12x - 45 \text{ where } 0 \le x \le 3$

Taking the derivative of f with respect to x, we have

$$\frac{df}{dx} = \frac{d}{dx}\left(x^3 - 12x - 45\right) = 3x^2 - 12$$

$$\frac{df}{dx} = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow 3\left(x^2 - 4\right) = 3(x - 2)(x + 2) = 0 \Rightarrow x = \pm 2$$

Thus, $x = -2,\ x = 2$ are the possible points of local extrema. Note that only $x = 2$ belongs to the given interval $0 \le x \le 3$.

Now, taking the second derivative of f with respect to *x*, we have

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(3x^2 - 12\right) = 6x$$

When $x = 2, d^2f/d^2x = 12 > 0$. So, $x = 2$ is a point of local minimum.

Next, we evaluate $f(x)$ at $x = 2$ and at the end points $x = 0, x = 3$.

$$f(0) = (0)^3 - 12(0) - 45 = -45$$

$$f(2) = (2)^3 - 12(2) - 45 = 8 - 24 - 45 = -61$$

$$f(3) = (3)^3 - 12(3) - 45 = 27 - 36 - 45 = -54$$

We find that $f(2) < f(3) < f(0)$. Hence, in the interval $0 \le x \le 3$, the absolute extrema are as follows.

The absolute maximum value of $f(x)$ is -45 which it attains at $x = 0$.
The absolute minimum value of $f(x)$ is -61, which it attains at $x = 2$.

Example 2: Find the points of local maxima or local minima and the corresponding maximum and minimum values the following function

$$f(x) = (x + 1)^2(x - 1)$$

Solution: We have $f(x) = (x + 1)^2(x - 1)$. First, we find $\frac{dy}{dx}$, for which we apply the product rule. Let

$$f(x) = (x + 1)^2 \quad g(x) = (x - 1) \quad \text{so that } \frac{d}{dx}(x + 1)^2(x - 1) = \frac{d}{dx}(fg) = ?$$

Applying the product rule, we have $\frac{dy}{dx} = \frac{d}{dx}(fg)$

$$= f\frac{dg}{dx} + g\frac{df}{dx} = (x + 1)^2\left[\frac{d}{dx}(x - 1)\right] + (x - 1)\left[\frac{d}{dx}(x + 1)^2\right]$$

$$= (x + 1)^2(1) + (x - 1)\left[\frac{d}{dx}(x^2 + 2x + 1)\right]$$

$$= (x + 1)^2 + (x - 1)(2x + 2) = (x + 1)^2 + 2(x - 1)(x + 1)$$
$$= (x + 1)[(x + 1) + 2(x - 1)\ ] = (x + 1)(x + 1 + 2x - 2\ ) = (x + 1)(3x - 1\ )$$
$$\frac{dy}{dx} = 0 \Rightarrow (x + 1)(3x - 1\ ) = 0 \Rightarrow x = -1, \frac{1}{3}$$

Thus, $x = -1, x = 1/3$ are the possible points of local extrema.

Now, taking the second derivative of y with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[(x + 1)(3x - 1)] = \frac{d}{dx}[x(3x - 1\ ) + 1(3x - 1)]$$

$$= \frac{d}{dx}[3x^2 - x + 3x - 1] = \frac{d}{dx}[3x^2 + 2x - 1] = 6x + 2$$

When $x = -1, d^2y/d^2x = 6(-1) + 2 = -4 < 0$. So, $x = -1$ is a point of local maximum.

When $x = 1/3$, $d^2y/d^2x = 6\left(\frac{1}{3}\right) + 2 = 4 > 0$. So, $x = 1/3$ is a point of local minimum.

The local maximum value of y at $x = -1$ is

$y(-1) = (-1 + 1)^2(-1 - 1) = 0$

The local minimum value of y at $x = 1/3$ is

$$y\left(\frac{1}{3}\right) = \left(\frac{1}{3} + 1\right)^2\left(\frac{1}{3} - 1\right) = \left(\frac{4}{3}\right)^2\left(- \frac{2}{3}\right) = \left(\frac{16}{9}\right)\left(- \frac{2}{3}\right) =- \frac{32}{27}$$

Chapter 12 Exercises

Find the following derivatives with respect to the stated variable.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1 Find the absolute extrema of the following function in the interval $[-3,3]$

$$f(x) = x^3 - 3x$$

2

Find the absolute extrema of the following function in the interval $[-2,3]$

$$f(x) = x^3 + x^2 - x$$

3 Find the absolute extrema of the following function in the interval $\left[\frac{1}{3},2\right]$

$$f(x) = \frac{3}{x^4} - \frac{2}{x^6}$$

Chapter 12 Solutions

1 $f(x) = x^3 - 3x, \quad -3 \le x \le 3$

Taking the derivative of f with respect to x, we have

$$\frac{df}{dx} = \frac{d}{dx}(x^3 - 3x) = 3x^2 - 3$$

$$\frac{df}{dx} = 0 \Rightarrow 3x^2 - 3 = 0$$

$$3(x^2 - 1) = 3(x - 1)(x + 1) = 0 \Rightarrow x = \pm 1$$

Thus, $x = -1, x = +1$ are the possible points of local extrema.

Now, taking the second derivative of f with respect to x, we have

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}(3x^2 - 3) = 6x$$

When $x = -1$, $d^2f/d^2x = -6 < 0$. So, $x = -1$ is a point of local maximum.

When $x = +1$, $d^2f/d^2x = +6 > 0$. So, $x = +1$ is a point of local minimum.

Next, we evaluate $f(x)$ at $x = -1, x = +1$ and at the end points $x = -3, x = +3$.

$$f(-3) = (-3)^3 - 3(-3) = -27 + 9 = -18$$

$$f(-1) = (-1)^3 - 3(-1) = -1 + 3 = +2$$

$$f(+1) = (+1)^3 - 3(+1) = +1 - 3 = -2$$

$$f(+3) = (+3)^3 - 3(+3) = +27 - 9 = +18$$

We find that

$$f(-3) < f(+1) < f(-1) < f(+3)$$

Hence, in the interval $-3 \le x \le 3$ the absolute extrema of $f(x)$ are as follows.

- The absolute maximum value of $f(x)$ is **18** which it attains at $x = +3$.
- The absolute minimum value of $f(x)$ is -18, which it attains at $x = -3$.

2 $f(x) = x^3 + x^2 - x, \quad -2 \le x \le 3$

Taking the derivative of f with respect to x, we have

$$\frac{df}{dx} = \frac{d}{dx}\left(x^3 + x^2 - x\right) = 3x^2 + 2x - 1$$

$$\frac{df}{dx} = 0 \Rightarrow 3x^2 + 2x - 1 = 0$$

$$3x^2 + 3x - x - 1 = 0$$

$$3x(x + 1) - (x + 1) = (3x - 1)(x + 1) = 0 \Rightarrow x = \frac{1}{3}, -1$$

Thus, $x = 1/3, x = -1$ are the possible points of local extrema.

Now, taking the second derivative of f with respect to x, we have

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(3x^2 + 2x - 1\right) = 6x + 2$$

When $x = \frac{1}{3}, d^2f/d^2x = 6\left(\frac{1}{3}\right) + 2 = 4 > 0$. So, $x = 1/3$ is a point of local minimum.

When $x = -1, d^2f/d^2x = 6(-1) + 2 = -4 < 0$. So, $x = -1$ is a point of local maximum.

Next, we evaluate $f(x)$ at $x = 1/3, x = -1$ and at the end points $x = -2, x = 3$.

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 - \frac{1}{3} = \frac{1}{27} + \frac{1}{9} - \frac{1}{3} = -\frac{5}{27}$$

$$f(-1) = (-1)^3 + (-1)^2 - (-1) = -1 + 1 + 1 = 1$$

$$f(-2) = (-2)^3 + (-2)^2 - (-2) = -8 + 4 + 2 = -2$$

$$f(3) = (3)^3 + (3)^2 - (3) = 27 + 9 - 3 = 33$$

We find that

$$f(-2) < f\left(\frac{1}{3}\right) < f(-1) < f(3)$$

Hence, in the interval $-2 \le x \le 3,$ the absolute extrema of $f(x)$ are as follows.

- The absolute maximum value of $f(x)$ is 33 which it attains at $x = 3$.
- The absolute minimum value of $f(x)$ is (-2), which it attains at $x = -2$.

3

$$f(x) = \frac{3}{x^4} - \frac{2}{x^6},\ \frac{1}{3} \le x \le 2$$

Taking the derivative of $f(x)$ with respect to x, we have

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{3}{x^4} - \frac{2}{x^6}\right) = \frac{d}{dx}\left(3x^{-4} - 2x^{-6}\right) = -12x^{-5} + 12x^{-7}$$

$$\frac{df}{dx} = 0 \Rightarrow = -12x^{-5} + 12x^{-7} = 0 \Rightarrow -\frac{12}{x^5} + \frac{12}{x^7} = 0 \Rightarrow \frac{12}{x^7} = \frac{12}{x^5}$$

Multiplying both sides by x^7, we have

$$\frac{x^7}{x^7} = \frac{x^7}{x^5} \Rightarrow 1 = x^2 \Rightarrow x = \pm 1$$

Thus, $x = 1,\ x = -1$ are the possible points of local extrema. Note that only $x = 1$ belongs to the given interval $\frac{1}{3} \le x \le 2$.

Now, taking the second derivative of f with respect to *x*, we have

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left[-12x^{-5} + 12x^{-7}\right] = 60x^{-6} - 84x^{-8}$$

When $x = 1, \frac{d^2f}{d^2x} = 60(1)^{-6} - 84(1)^{-8} = -24 < 0.$ So, $x = 1$ is a point of local maximum.

Next, we evaluate $f(x)$ at $x = 1$ and at the end points $x = \frac{1}{3}$ and $x = 2$.

$$f(1) = \frac{3}{1^4} - \frac{2}{1^6} = 3 - 2 = 1$$

$$f\left(\frac{1}{3}\right) = \frac{3}{(1/3)^4} - \frac{2}{(1/3)^6} = 3(3)^4 - 2(3)^6 = (3)^4\left[3 - 2(3)^2\right] = 81[3 - 18]$$

$$= -81 \times 15 = -1215$$

$$f(2) = \frac{3}{2^4} - \frac{2}{2^6} = \frac{1}{2^4}\left(3 - \frac{2}{2^2}\right) = \frac{1}{2^4}\left(3 - \frac{1}{2}\right) = \frac{1}{16}\left(\frac{5}{2}\right) = \frac{5}{32}$$

We find that $f\left(\frac{1}{3}\right) < f(2) < f(1)$

Hence, in the interval $\frac{1}{3} \le x \le 2$, the absolute extrema are as follows.

The absolute maximum value of $f(x)$ is 1 which it attains at $x = 1$.

The absolute minimum value of $f(x)$ is (-1215), which it attains at $x = 1/3$.

13 LIMITS AND L'HÔPITAL'S RULE

We start by giving an informal idea of limit of a function. We say that l is the limit of the function $f(x)$ at *c* if $f(x)$ moves closer and closer to l when x approaches *c*. We write this symbolically as

$$\lim_{x \to c} f(x) = l$$

Note that *x* can approach *c* in two ways. Either all the values of *x* near *c* could be less than or could be greater than c. Correspondingly, there are the following two limits. The *left hand limit* is the value that $f(x)$ approaches as *x* approaches *c* from the left, that is, it starts with values of $x < c$ and then moves closer to *c*. We use the notation $\lim_{x \to c^-} f(x)$ for the left hand limit. The *right hand limit* is the value that $f(x)$ approaches as *x* approaches *c* from the right, that is, it starts with values of $x > c$ and then moves closer to *c*. We use the notation $\lim_{x \to c^+} f(x)$ for the right hand limit. When the left hand limit and the right hand limits coincide, we call the common value as the limit of $f(x)$ at *c*. Thus

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)$$

For finding $\lim_{x \to c} f(x),$ we should first try substituting $x = c$ in $f(x)$. If we get a finite value of $f(c),$ then this finite value is the required limit.

Now consider evaluation of $\lim_{x \to c} \frac{f(x)}{g(x)},$ involving quotient of two functions. When both $f(x)$ and $g(x)$ become zero or both grow to infinity, $\lim_{x \to c} \frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (known as indeterminate forms), we use the following techniques. In cases of indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$ involving algebraic functions, we can use the following methods:

(i) factorization method,

(ii) the rationalization method and

(iii) the method of division of each term of numerator and denominator by the highest power of x in the numerator or denominator.

Study carefully the following solved examples to see how the above methods can be used to find the limits of algebraic functions.

L'HÔPITAL'S RULE

For indeterminate forms, particularly those involving trigonometric, inverse trigonometric, exponential and logarithmic functions $\lim_{x \to c} \frac{f(x)}{g(x)}$ can be evaluated by the use of l'Hôpital's rule. This rule states that

$$\text{If } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \qquad \text{or} \qquad \text{If } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\ \infty, \text{ then}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{\frac{df}{dx}}{\frac{dg}{dx}}$$

In other words, l'Hôpital's rule states that when $\lim_{x \to c} \frac{f(x)}{g(x)}$ is indeterminate, we can evaluate it by obtaining the limit of the quotient of the derivatives of $f(x)$ and $g(x)$. Note that we can try calculating the limits of derivatives by substituting $x = c$ in $\frac{df}{dx}$ and $\frac{dg}{dx}$. If we again encounter indeterminate form $\frac{0}{0}$, we can repeat the above process, till we obtain a finite value.

Example 1: Evaluate the following limit

$$\lim_{x \to \infty} \frac{3x^2 - 2x - 3}{x^2 - 4x - 2}$$

Solution: When x grows larger and larger, both the numerator and the denominator approach infinity. As a result, the algebraic expression $\frac{3x^2 - 2x - 3}{x^2 - 4x - 2}$ assumes the indeterminate form $\frac{\infty}{\infty}$ form. In this case, we prefer the method of division of each term of numerator and denominator by x^2, which is the highest power of x in the numerator or denominator. Note that we could also have applied the l'Hôpital's rule because of $\frac{\infty}{\infty}$ form.

$$\lim_{x \to \infty} \frac{3x^2 - 2x - 3}{x^2 - 4x - 2} = \lim_{x \to \infty} \frac{\frac{3x^2}{x^2} - \frac{2x}{x^2} - \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{4x}{x^2} - \frac{2}{x^2}} = \lim_{x \to \infty} \frac{3 - \frac{2}{x} - \frac{3}{x^2}}{1 - \frac{4}{x} - \frac{2}{x^2}}$$

$$= \frac{3 - 0 - 0}{1 - 0 - 0} = 3$$

Example 2: Evaluate the following limits

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4}$$

Solution: When we substitute $x = 2$, the expression $\frac{x^2 - 5x + 6}{x^2 - 4}$ assumes the indeterminate form $\frac{0}{0}$ since both the numerator and the denominator become 0. We can use l'Hôpital's rule to evaluate the limit. However, we prefer the algebraic method of factorization since we are dealing with algebraic expression.

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} = \lim_{x \to 2} \frac{x^2 - 2x - 3x + 6}{x^2 - 4} = \lim_{x \to 2} \frac{x(x - 2) - 3(x - 2)}{x^2 - 4}$$

$$= \lim_{x \to 2} \frac{(x - 2)(x - 3)}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{(x - 3)}{(x + 2)} = \frac{2 - 3}{2 + 2} = -\frac{1}{4}$$

Example 3: Evaluate the following limit

$$\lim_{x \to 0} \frac{x}{\tan x}$$

Solution: Substituting $x = 0$ we get $\lim_{x \to 0} \frac{x}{\tan x} = \frac{0}{\tan 0} = \frac{0}{0}$

Now, $\left(\frac{0}{0}\right)$ is indeterminate form. In this case, it is necessary to apply l'Hopital's rule since we are dealing with trigonometric function.

$$\lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(\tan x)} = \lim_{x \to 0} \frac{(1)}{\sec^2 x} = \lim_{x \to 0} \left(\cos^2 x\right) = \cos^2 0 = (1)^2 = 1$$

Example 4: Evaluate the following limit

$$\lim_{x \to 0} \frac{e^x - 1 - x}{3x}$$

Solution: Substituting $x = 0$ we get $\lim_{x \to 0} \frac{e^x - 1 - x}{3x} = \frac{e^0 - 1 - 0}{3(0)} = \frac{1 - 1}{0} = \frac{0}{0}$

Now, $\left(\frac{0}{0}\right)$ is indeterminate form. Since we are dealing with exponential function, we apply l'Hopital's rule to get the value of the limit.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{3x} = \lim_{x \to 0} \frac{\frac{d}{dx}\left(e^x - 1 - x\right)}{\frac{d}{dx}(3x)} = \lim_{x \to 0} \frac{e^x - 0 - 1}{3} = \frac{e^0 - 1}{3} = \frac{1 - 1}{3} = \frac{0}{3} = 0$$

Chapter 13 Exercises

Set A

Evaluate the following limits.

You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\lim_{x \to 2} \frac{x^2 - 3}{x + 1} =$$

2

$$\lim_{x \to 9} \frac{x^2 - 81}{x - 9} =$$

3

$$\lim_{x \to \infty} \frac{5x^4 - 2x^2 - 3}{x^5 - 3x^2 - 2} =$$

4

$$\lim_{x \to a} \frac{\sqrt{x - a}}{x + a} =$$

5

$$\lim_{x \to \pi/2} x \sin x =$$

6

$$\lim_{x \to 3} \frac{\sqrt{4 - x} - 1}{3 - x} =$$

Set B

7

$$\lim_{x \to \infty} \frac{4x^2 - 5x}{9x^2 - 2} =$$

8

$$\lim_{x\to 0}\frac{\ln(1+3x)}{e^x}=$$

[redacted]

$$\lim_{x\to 3}\frac{\sqrt{3x+2}}{x+1}=$$

10

$$\lim_{x\to\infty}\frac{x^3+x^2-2}{3x^3+x-4}=$$

11

$$\lim_{x\to\pi/2}\frac{4x-2\pi}{\cos x}=$$

12

$$\lim_{x\to 0}\frac{\sin 3x}{x}=$$

Set C

13

$$\lim_{x\to\infty}\frac{3\ln x}{2x}$$

14

$$\lim_{x\to 0}\frac{e^x-1}{x}=$$

15

$$\lim_{x\to 0}\frac{e^{3x}-1}{\sin x}=$$

16

$$\lim_{x\to 5}\frac{e^x-e^5}{x-5}$$

Chapter 13 Solutions

Set A

1

Substituting $x = 2$, we get $\lim\limits_{x\to 2}\frac{x^2 - 3}{x + 1} = \frac{2^2 - 3}{2 + 1} = \frac{1}{3}$.

Explanation: Recall that if by direct substitution of $x = 2$, we get a finite number, then the finite number is the value of the limit.

When we substitute $x = 9$, the expression $\frac{x^2 - 81}{x - 9}$ assumes the indeterminate form $\frac{0}{0}$ form since both the numerator and the denominator become 0. Let us, therefore, use the algebraic method of factorization.

$$\lim_{x\to 9}\frac{x^2 - 81}{x - 9} = \lim_{x\to 9}\frac{(x + 9)(x - 9)}{(x - 9)} = \lim_{x\to 9}(x + 9) = 9 + 9 = 18$$

Explanation: Since we get indeterminate form $\frac{0}{0}$, we could have used l’Hôpital’s rule to evaluate the limit. However, we prefer the algebraic method where it is applicable.

When we substitute $x = \infty$, the expression $\frac{5x^4 - 2x^2 - 3}{x^5 - 3x^2 - 2}$ assumes the indeterminate form $\frac{\infty}{\infty}$ form since both the numerator and the denominator approach ∞. Let us, therefore, use the algebraic method of division of each term of numerator and denominator by the highest power of x in the numerator or denominator.

$$\lim_{x\to\infty}\frac{5x^4 - 2x^2 - 3}{x^5 - 3x^2 - 2} = \lim_{x\to\infty}\frac{\frac{5x^4}{x^5} - \frac{2x^2}{x^5} - \frac{3}{x^5}}{\frac{x^5}{x^5} - \frac{3x^2}{x^5} - \frac{2}{x^5}} = \lim_{x\to\infty}\frac{\frac{5}{x} - \frac{2}{x^3} - \frac{3}{x^5}}{1 - \frac{3}{x^3} - \frac{2}{x^5}}$$

$$= \frac{0 - 0 - 0}{1 - 0 - 0} = \frac{0}{1} = 0$$

Explanation:

- Since we get indeterminate form $\frac{\infty}{\infty}$, we could have used l’Hôpital’s rule to evaluate the limit. However, we prefer the algebraic method where it is applicable. $\frac{5}{x}, \frac{2}{x^3}, \frac{3}{x^5}, \frac{3}{x^3}$ and $\frac{2}{x^5}$, all approach 0 when x grows infinitely large.
- We can check the answer by using the calculator. If we take sufficiently large x, say, x

=1000, the expression $\frac{5x^4 - 2x^2 - 3}{x^5 - 3x^2 - 2} = \frac{4.999997 \times 10^{12}}{1000.0000 \times 10^{12}} \approx 0.005 \approx 0.$

4

Substituting $x = a$ we get

$$\lim_{x \to a} \frac{\sqrt{x - a}}{x + a} = \frac{\sqrt{a - a}}{a + a} = 0$$

Explanation:

- Since, by substituting $x = 0$, we get a finite number 0, the value of the limit is 0.

5

Substituting $x = \pi/2$ we get

$$\lim_{x \to \pi/2} x \sin x = \frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2}(1) = \frac{\pi}{2}$$

Explanation:

- Since, by substituting $x = \frac{\pi}{2}$, we get a finite number $\frac{\pi}{2}$, the value of the limit is $\frac{\pi}{2}$.
- Recall that $\sin \frac{\pi}{2} = \sin 90^0 = 1.$

When we substitute $x = 3$, the expression $\frac{\sqrt{4 - x} - 1}{3 - x}$ assumes the indeterminate form $\frac{0}{0}$ form since both the numerator and the denominator approach 0. Let us, therefore, use the algebraic method of rationalization.

$$\lim_{x \to 3} \frac{\sqrt{4 - x} - 1}{3 - x} = \lim_{x \to 3} \frac{(\sqrt{4 - x} - 1)(\sqrt{4 - x} + 1)}{(3 - x) \quad (\sqrt{4 - x} + 1)} = \lim_{x \to 3} \frac{\left[(\sqrt{4 - x})^2 - (1)^2\right]}{(3 - x)(\sqrt{4 - x} + 1)}$$

$$= \lim_{x \to 3} \frac{(4 - x - 1)}{(3 - x)(\sqrt{4 - x} + 1)} = \lim_{x \to 3} \frac{(3 - x)}{(3 - x)(\sqrt{4 - x} + 1)} = \lim_{x \to 3} \frac{1}{\sqrt{4 - x} + 1}$$

$$= \frac{1}{\sqrt{4 - 3} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

Explanation:

- Since we get indeterminate form $\frac{0}{0}$, we could have used l’Hôpital’s rule to evaluate the limit. However, we prefer the algebraic method where it is applicable.
- We have used the formula $(a - b)(a + b) = a^2 - b^2$, with $a = \sqrt{4 - x}$, $b = 1$.
- We can check the answer by using the calculator. If we take x sufficiently close to 3,

say, x=2.9, the expression $\frac{\sqrt{4 - x} - 1}{3 - x} = \frac{0.049}{0.1} \approx 0.49 \approx 0.5 = \frac{1}{2}$

Set B

When we substitute $x = \infty$, the expression $\lim\limits_{x \to \infty} \frac{4x^2 - 5x}{9x^2 - 2}$ assumes the indeterminate form $\frac{\infty}{\infty}$ form since both the numerator and the denominator approach ∞. Let us, therefore, use the algebraic method of division of each term of numerator and denominator by the highest power of x in the numerator or denominator.

$$\lim_{x \to \infty} \frac{4x^2 - 5x}{9x^2 - 2} = \lim_{x \to \infty} \frac{\frac{4x^2}{x^2} - \frac{5x}{x^2}}{\frac{9x^2}{x^2} - \frac{2}{x^2}} = \lim_{x \to \infty} \frac{4 - \frac{5}{x}}{9 - \frac{2}{x^2}} = \frac{4 - 0}{9 - 0} = \frac{4}{9}$$

Explanation:

- Since we get indeterminate form $\frac{\infty}{\infty}$, we could have used l'Hôpital's rule to evaluate the limit. However, wherever applicable, we prefer the algebraic method.
- $\frac{5}{x}$ and $\frac{2}{x^2}$ both approach 0 when x grows infinitely large.
- We can check the answer by using the calculator. If we take sufficiently large x -say, x =1000, the expression $\frac{4x^2 - 5x}{9x^2 - 2} = \frac{3995000}{8999998} \approx 0.444 \approx \frac{4}{9}$.

Substituting $x = 0$ we get

$$\lim_{x \to 0} \frac{\ln(1 + 3x)}{e^x} = \frac{\ln(1 + 0)}{e^0} = \frac{\ln(1)}{e^0} = \frac{0}{1} = 0$$

Explanation:

Since, by substituting $x = 0$, we get a finite number 0, the value of the limit is 0.

Substituting $x = 3$ we get

$$\lim_{x \to 3} \frac{\sqrt{3x + 2}}{x + 1} = \frac{\sqrt{3(3) + 2}}{3 + 1} = \frac{\sqrt{11}}{4}$$

Explanation:

- Since, by substituting $x = 3$, we get a finite number $\frac{\sqrt{11}}{4}$, the value of the limit is $\frac{\sqrt{11}}{4}$.

10 When we substitute $x = \infty$, the expression $\frac{x^3 + x^2 - 2}{3x^3 + x - 4}$ assumes the indeterminate form $\frac{\infty}{\infty}$ form since both the numerator and the denominator approach ∞. Let us, therefore, use the algebraic method of division of each term of numerator and denominator by the highest power of x in the numerator or denominator.

$$\lim_{x\to\infty}\frac{x^3 + x^2 - 2}{3x^3 + x - 4} == \lim_{x\to\infty}\frac{\frac{x^3}{x^3} + \frac{x^2}{x^3} - \frac{2}{x^3}}{3\frac{x^3}{x^3} + \frac{x}{x^3} - \frac{4}{x^3}} = \lim_{x\to\infty}\frac{1 + \frac{1}{x} - \frac{2}{x^3}}{3 + \frac{1}{x^2} - \frac{4}{x^3}}$$

$$= \frac{1 + 0 - 0}{3 - 0 - 0} = \frac{1}{3}$$

Explanation:

- Since we get indeterminate form $\frac{\infty}{\infty}$, we could have used l'Hôpital's rule to evaluate the limit. However, wherever applicable, we prefer the algebraic method.
- $\frac{1}{x}, \frac{2}{x^3}, \frac{1}{x^2} \text{and} \frac{4}{x^3}$, all approach 0 when x grows infinitely large.
- We can check the answer by using the calculator. If we take sufficiently large x, say, x =1000, the expression $\frac{x^3 + x^2 - 2}{3x^3 + x - 4} = \frac{1000999998}{3000000996} \approx 0.334 \approx \frac{1}{3}$.

11

Substituting $x = \pi/2$ we get

$$\lim_{x\to\pi/2}\frac{4x - 2\pi}{\cos x} = \frac{4\left(\frac{\pi}{2}\right) - 2\pi}{\cos\frac{\pi}{2}} = \frac{2\pi - 2\pi}{0} = \frac{0}{0}$$

Now, $\left(\frac{0}{0}\right)$ is indeterminate form. Therefore, we apply l'Hopital's rule to get

$$\lim_{x\to\pi/2}\frac{4x-2\pi}{\cos x} = \lim_{x\to\pi/2}\frac{\frac{d}{dx}(4x-2\pi)}{\frac{d}{dx}(\cos x)} = \lim_{x\to\pi/2}\frac{\frac{d}{dx}(4x)-\frac{d}{dx}(2\pi)}{(-\sin x)} = \lim_{x\to\pi/2}\frac{4-0}{(-\sin x)}$$

$$= \frac{4}{\left(-\sin\frac{\pi}{2}\right)} = \frac{4}{-1} = -4$$

Explanation:

- Recall that $\cos\frac{\pi}{2} = \cos 90^0 = 0$ and $\sin\frac{\pi}{2} = \sin 90^0 = 1.$
- We can check the answer by using the calculator. If we take x sufficiently close to $\frac{\pi}{2}(\approx 1.571)$, say, x=1.6, the expression $\frac{4x-2\pi}{\cos x} = \frac{0.116815}{-0.0292} \approx -4.0005 \approx -4$.

12 Substituting $x = 0$ we get

$$\lim_{x\to 0}\frac{\sin 3x}{x} = \frac{\sin[3(0)]}{0} = \frac{\sin 0}{0} = \frac{0}{0}$$

Now, $\left(\frac{0}{0}\right)$ is indeterminate form. Therefore, we apply l’Hopital’s rule to get

$$\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}(\sin 3x)}{\frac{d}{dx}(x)}$$

We will apply the chain rule to find $\frac{d}{dx}(\sin 3x)$. Let $y = \sin 3x$

Put $u = 3x$, so that $y(u) = \sin u,\qquad u(x) = 3x,\qquad \frac{d}{dx}\sin 3x = \frac{dy}{dx} = ?$

Applying the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}(\sin u)\right]\left[\frac{d}{dx}(3x)\right] = (\cos u)\,(3) = 3\cos u = 3\cos 3x$$

Therefore, $\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}(\sin 3x)}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{3\cos 3x}{1} = \frac{3\cos[3(0)]}{1}$

$= 3\cos 0 = 3(1) = 3$

Explanation: Recall that $\sin 0 = 0,\ \cos 0 = 1.$

- We can check the answer by using the calculator. If we take x sufficiently close to 0 ,

say $x = 0.001$, the expression $\frac{\sin 3x}{x} \approx \frac{0.003}{0.001} = 3$.

Set C

When we substitute $x = \infty$, the expression $\frac{3\ln x}{2x}$ assumes the indeterminate form $\frac{\infty}{\infty}$ form since both the numerator and the denominator approach ∞.
Therefore, we apply l'Hopital's rule to get

$$\lim_{x\to\infty}\frac{3\ln x}{2x} = \lim_{x\to\infty}\frac{\frac{d}{dx}(3\ln x)}{\frac{d}{dx}(2x)} = \lim_{x\to\infty}\frac{\left(\frac{3}{x}\right)}{2} = \frac{0}{2} = 0$$

Explanation:

- Recall that $\frac{d}{dx}(3\ln x) = \frac{3}{x}$ and $\frac{d}{dx}(2x) = 2$
- We can check the answer by using the calculator. If we take sufficiently large x, say, x =100,000, the expression $\frac{3\ln x}{2x} = \frac{34.453887764}{200{,}000} \approx 0.0001727 \approx 0.$

14 Substituting $x = 0$ we get

$$\lim_{x\to 0}\frac{e^x - 1}{x} = \frac{e^0 - 1}{0} = \frac{1 - 1}{0} = \frac{0}{0}$$

Now, $\left(\frac{0}{0}\right)$ is indeterminate form. Therefore, we apply l'Hopital's rule to get

$$\lim_{x\to 0}\frac{e^x - 1}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{\frac{d}{dx}(e^x) - \frac{d}{dx}(1)}{1} = \lim_{x\to 0}\frac{e^x - 0}{1} = \frac{1 - 0}{1} = 1$$

Explanation: Recall that $e^0 = 1$, $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}x = 1$.

- We can check the answer by using the calculator. If we take $x = 0.001$, the expression $\frac{e^x - 1}{0.001} \approx \frac{0.0010005}{0.001} \approx 1.0005 \approx 1$.

15

Substituting $x = 0$ we get

$$\lim_{x\to 0}\frac{e^{3x} - 1}{\sin x} = \frac{e^{3(0)} - 1}{\sin 0} = \frac{e^0 - 1}{0} = \frac{1 - 1}{0} = \frac{0}{0}$$

Now, $\left(\frac{0}{0}\right)$ is indeterminate form. Therefore, we apply l'Hopital's rule to get

$$\lim_{x\to 0}\frac{e^{3x}-1}{\sin x} = \lim_{x\to 0}\frac{\frac{d}{dx}\left(e^{3x}-1\right)}{\frac{d}{dx}(\sin x)} = \lim_{x\to 0}\frac{\frac{d}{dx}\left(e^{3x}\right)-\frac{d}{dx}(1)}{\cos x} = \lim_{x\to 0}\frac{3e^{3x}-0}{\cos x}$$

$$= \lim_{x\to 0}\frac{3e^{3x}}{\cos x} = \frac{3e^{3(0)}}{\cos 0} = \frac{3e^{0}}{1} = \frac{3(1)}{1} = 3$$

Explanation: Recall that $e^{0} = 1,\ \sin 0 = 0.\ \frac{d}{dx}\left(e^{3x}\right) = 3e^{3x}\ \cos 0^{0} = 1.$

- We can check the answer by using the calculator. If we take, $x = 0.001$, the expression $\frac{e^{3x}-1}{\sin x} \approx \frac{0.0030045}{0.0009999} \approx 3.00480 \approx 3.$

16 Substituting $x = 5$ we get $\lim_{x\to 5}\frac{e^{x}-e^{5}}{x-5} = \frac{e^{5}-e^{5}}{5-5} = \frac{0}{0}.$

Now, $\left(\frac{0}{0}\right)$ is indeterminate form. Therefore, we apply l'Hopital's rule to get

$$\lim_{x\to 5}\frac{e^{x}-e^{5}}{x-5} = \lim_{x\to 5}\frac{\frac{d}{dx}\left(e^{x}-e^{5}\right)}{\frac{d}{dx}(x-5)} = \lim_{x\to 5}\frac{e^{x}-0}{1-0} = \lim_{x\to 5}\frac{e^{x}}{1} = \frac{e^{5}}{1} = e^{5} \approx 148.41$$

Explanation: Recall that $\frac{d}{dx}e^{x} = e^{x}$ and $\frac{d}{dx}x = 1$.We can check the answer by using the calculator. If we plug in $x = 5.001$, the expression $\frac{e^{x}-e^{5}}{x-5} = \frac{1.491577}{0.001} = \frac{0.1484873}{0.001} = 148.49 \approx e^{5}.$

14 INDEFINITE INTEGRALS AND INTEGRALS OF POLYNOMIALS

We start by introducing the concept of indefinite integral of a function. Suppose we are given a function $f(x)$, which is the derivative of the function $F(x)$ with respect to x, that is $f(x) = \frac{d}{dx}F(x)$, then the function $F(x)$ is called the antiderivative (or integral) of the function $f(x)$ The process of finding antiderivatives is called integration. We find that the integration is the inverse of differentiation. Here, instead of finding the derivative of a function, we are required to find the function $F(x)$ that could possibly have given function $f(x)$ as a derivative.

Note that the derivative of any constant c is zero. If $F(x)$ is antiderivative, then $F(x) + c$ is also antiderivative. We use the symbol $\int f(x)\, dx$ to represent the antiderivative. In this notation, $f(x)$ is called the integrand.

$$f(x) = \frac{d}{dx}[F(x) + c] \leftrightarrow \int f(x)\, dx = F(x) + c$$

The general antiderivative with arbitrary constant c is known as the indefinite integral of $f(x)$. As an example, we know that $\frac{d}{dx}(x^5 + c) = 5\,x^4$, that is, $5\,x^4$ is the derivative of $x^5 + c$. We say that $x^5 + c$ is the indefinite integral of $5\,x^4$.

$$5\,x^4 = \frac{d}{dx}(x^5 + c) \leftrightarrow \int 5\,x^4\, dx = x^5 + c$$

We now discuss how to find the indefinite integrals by using a formula, which is known as power rule of integration.

Consider a term of the form ax^n, where a is a constant coefficient and the exponent *n* is a real number: zero, positive, negative or a fraction. Then the indefinite integral of ax^n equals ax^{n+1} divided by $n + 1$ with arbitrary constant as per the following formula:

$$\int ax^n\, dx = \frac{ax^{n+1}}{n + 1} + c \quad \text{if } n \neq -1$$

$$\int ax^{-1}\, dx = \int \frac{a}{x}\, dx = a\ln x + c \qquad \text{when } n = -1$$

As in case of derivatives, the above formula for integral of ax^n is valid for n being zero, positive number, negative number or a fraction. As such, it is applicable not only to the case of

polynomials but also to more general terms having power of x being negative or a fraction.

Here, it is important to know the following two basic rules of integration that are applicable to all types of functions including polynomial functions.

1. The first of these rules is quite similar to the constant multiple rule of differentiation and may, therefore, be called constant multiple rule for integration. This rule states that that the integral of constant times a function $f(x)$ is constant times the integral of the function:

$$\int af(x)\, dx = a\int f(x)\, dx$$

For example,

$$\int 3\sin x\, dx = 3\int \sin x\, dx$$

2. The second rule is quite similar to the addition rule of differentiation and may, therefore, be called addition rule for integration. This rule is also applicable to all types of function. It states that the integral of the sum of a number of functions equals the sum of integrals of these functions. Thus, if $f_1, f_2, f_3,$ …, f_n are functions of x, then

$$\int (f_1 + f_2 + f_3 + \ldots + f_n)\, dx = \int f_1\, dx + \int f_2\, dx + \int f_3\, dx + \ldots \int f_n\, dx$$

As an example, suppose we have a polynomial $f(x) = 5x^4 - 3x^3 + 4$ having three terms. Let $f_1 = 5x^4,\ f_2 =- 3x^3$ and $f_3 = 4$, then $5x^4 + (-3x^3) + 4 = 5x^4 - 3x^3 + 4.$ Application of the addition rule gives

$$\int (5x^4 - 3x^3 + 4)\, dx = \int 5x^4\, dx - \int 3x^3\, dx + \int 4\, dx$$

Similarly,

$$\int (\sin x + \tan x)\, dx = \int \sin x\, dx + \int \tan x\, dx$$

We now recall some results of exponents of real numbers, which we will often make use of.

$$x^m x^n = x^{m+n}, \qquad \frac{1}{x^n} = x^{-n}, \qquad \frac{x^m}{x^n} = x^m x^{-n} = x^{m-n},$$

$$(x^m)^n = x^{mn} \qquad (cx)^n = c^n x^n, \qquad \sqrt{cx} = (cx)^{1/2} = c^{1/2}x^{1/2} = \sqrt{c}\sqrt{x}$$

$$x^{m/n} = (x^{1/n})^m = (\sqrt[n]{x})^m = (x^m)^{1/n} = \sqrt[n]{x^m}$$

- Note that when no multiplying coefficient or exponent is explicitly written, the same equals 1. Thus, for example, $x^2 = 1x^2$ and $x = 1x^1$.
- It is also useful to note the following results

$$x^0 = 1,\ \frac{1}{x} = \frac{1}{x^1} = x^{-1},\ \sqrt{x} = x^{1/2}, \qquad \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$$

Example 1: Evaluate the following integral

$$\int 5x^4 \, dx$$

Solution: Comparing $5x^4$ with the general term ax^n, we find that $a = 5$ and $n = 4$. Applying the formula $\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$, we find that

$$\int 5x^4 \, dx = \frac{5x^{4+1}}{4+1} + c = x^5 + c$$

Check: $\frac{d}{dx}(x^5 + c) = \frac{d}{dx}(x^5) + \frac{d}{dx}(c) = 5x^4 + 0 = 5x^4$

Example 2: Evaluate the following integral

$$\int \frac{7}{x^5} \, dx$$

Solution: Using the result $\frac{7}{x^5} = 7x^{-5}$ and comparing $7x^{-5}$ with ax^n, we find that $a = 7$ and $n = -5$. Use of formula $\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$ gives

$$\int \frac{7}{x^5} \, dx = \int 7x^{-5} dx = \frac{7x^{-5+1}}{-5+1} + c = -\frac{7}{4}(x^{-4}) + c = -\frac{7}{4x^4} + c$$

Check: $\frac{d}{dx}\left(-\frac{7}{4}x^{-4} + c\right) = \left(-\frac{7}{4}\right)(-4)x^{-4-1} + 0 = 7x^{-5} = \frac{7}{x^5}$

Example 3: Evaluate the following integral

$$\int \frac{4}{x} \, dx$$

Solution: Using the result $\frac{4}{x} = 4x^{-1}$ and comparing $4x^{-1}$ with ax^n, we find that $a = 4$ and $n = -1$. For special case of $n = -1$, use of formula $\int ax^{-1} \, dx = a\ln x + c$ gives

$$\int \frac{4}{x} \, dx = 4\ln x + c$$

Check: $\frac{d}{dx}(4\ln x + c) = \frac{4}{x} + 0 = \frac{4}{x}$

Example 4: Evaluate the following integral

$$\int x^{2/3}\ dx$$

Solution: Comparing $x^{2/3}$ with ax^n, we find that $a = 1$ and $n = 2/3$. Use of formula $\int ax^n\ dx = \frac{ax^{n+1}}{n+1} + c$ gives

$$\int x^{2/3}\ dx = \frac{1x^{2/3+1}}{2/3+1} + c = \frac{x^{5/3}}{5/3} + c = \frac{3x^{5/3}}{5} + c$$

Check: Let $y = \frac{3x^{5/3}}{5} + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x^{5/3}}{5} + c\right) = \frac{d}{dx}\left(\frac{3x^{5/3}}{5}\right) + \frac{d}{dx}(c) = \left(\frac{3}{5}\right)\left(\frac{5}{3}x^{\frac{5}{3}-1}\right) + 0 = x^{2/3}$$

Example 5: Evaluate the following integral

$$\int (3x^3 + 2x^2)\ dx$$

Solution: In this case, we will use the addition rule of integration, which says that for functions f_1 and f_2,

$$\int (f_1 + f_2)\ dx = \int f_1\ dx + \int f_2\ dx$$

Substituting $f_1 = 3x^3,\ f_2 = 2x^2$, we have

$$\int (3x^3 + 2x^2)\ dx = \int 3x^3\ dx + \int 2x^2\ dx$$

We evaluate two antiderivatives $\int 3x^3\ dx$ and $\int 2x^2\ dx$ and add the two as follows.

$$\int (3x^3 + 2x^2)\ dx$$
$$= \int 3x^3\ dx + \int 2x^2\ dx = \frac{3x^{3+1}}{3+1} + \frac{2x^{2+1}}{2+1} + c = \frac{3x^4}{4} + \frac{2x^3}{3} + c$$

Check: Let $y = \frac{3x^4}{4} + \frac{2x^3}{3} + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x^4}{4}\right) + \frac{d}{dx}\left(\frac{2x^3}{3}\right) + \frac{d}{dx}(c) = \left(\frac{3}{4}\right)(4)x^{4-1} + \left(\frac{2}{3}\right)(3)x^{3-1} + 0 = 3x^3 + 2x^2$$

Example 6: Evaluate the following integral

$$\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$

Solution: In this case, we will use the addition rule of integration, which says that for functions f_1 and f_2,

$$\int(f_1 + f_2)\,dx = \int f_1\,dx + \int f_2\,dx$$

Substituting $f_1 = \sqrt{x},\ \ f_2 = \frac{1}{\sqrt{x}}$, we have

$$\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\ dx = \int\sqrt{x}\,dx + \int\frac{1}{\sqrt{x}}\,dx$$

We evaluate two antiderivatives $\int\sqrt{x}\,dx$ and $\int\frac{1}{\sqrt{x}}\,dx$ and add the two as follows.

$$\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\ dx = \int\sqrt{x}\,dx + \int\frac{1}{\sqrt{x}}\,dx$$

$$= \int 1x^{1/2}\,dx + \int 1x^{-1/2}\,dx = \frac{1x^{1/2+1}}{1/2+1} + \frac{1x^{(-1/2+1)}}{-1/2+1} + c$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + c = \frac{2}{3}x^{3/2} + 2x^{1/2} + c = \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + c$$

Note that we have written $x^{3/2} = x\,x^{1/2} = x\sqrt{x}$.

Check: Let $y = \frac{2}{3}x^{3/2} + 2x^{1/2} + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c\right) = \frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + \frac{d}{dx}\left(2x^{\frac{1}{2}}\right) + \frac{d}{dx}(c)$$

$$= \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{\left(\frac{3}{2}-1\right)} + (2)\left(\frac{1}{2}\right)x^{\left(\frac{1}{2}-1\right)} + 0$$

$$= x^{1/2} + x^{-1/2} = \sqrt{x} + \frac{1}{\sqrt{x}}$$

Chapter 14 Exercises

Set A

Evaluate the following integrals.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$\int 10x^{10}\, dx =$

2

$\int 15u^{7}\, du =$

3

$\int \frac{45}{x^{6}}\, dx =$

4

$\int \frac{3}{x}\, dx =$

5

$\int 5x^{4/7}\, dx =$

Set B

6

$\int 6\, dt =$

7

$\int 4x^{-3}\, dx =$

8

$\int 2xdx$

$$\int \frac{3}{x^{3/4}}\, dx =$$

Set C

10

$$\int (5x^5 + 10x^9)\, dx =$$

11

$$\int \left(3x^3 + 7 + \frac{4}{x}\right) dx =$$

12

$$\int (3x^{2/3} + 6x^{1/3})\, dx =$$

13

$$\int \left(\frac{3x^3 + 5x^2 + 8}{x^2}\right) dx =$$

Chapter 14 Solutions

Set A

1 We have

$$\int 10x^{10}\,dx = \frac{10x^{10+1}}{10+1} + c = \frac{10x^{11}}{11} + c$$

Check the answer: Let $y = \frac{10x^{11}}{11} + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{10x^{11}}{11} + c\right) = \frac{d}{dx}\left(\frac{10x^{11}}{11}\right) + \frac{d}{dx}(c) = \left(\frac{10}{11}\right)(11)x^{10} + 0 = 10x^{10}$$

2

We have

$$\int 15u^{7}\,du = \frac{15u^{7+1}}{7+1} + c = \frac{15u^{8}}{8} + c = \frac{15}{8}u^{8} + c$$

Check the answer: Let $y = \frac{15}{8}u^{8} + c,$ then

$$\frac{dy}{du} = \frac{d}{du}\left(\frac{15}{8}u^{8}\right) + \frac{d}{du}(c) = \left(\frac{15}{8}\right)(8)u^{8-1} + 0 = 15u^{7}$$

3

We have

$$\int \frac{45}{x^{6}}\,dx$$

$$= \int 45\,x^{-6}\,dx = \frac{45x^{-6+1}}{-6+1} + c = \frac{45x^{-5}}{-5} + c = -9x^{-5} + c = -\frac{9}{x^{5}} + c$$

Check the answer: Let $y = \; = -\frac{9}{x^{5}} + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{9}{x^{5}} + c\right) = (-9)(-5)x^{-5-1} + 0 = 45x^{-6} = \frac{45}{x^{6}}$$

Explanation:

- Note that $-6 + 1 = -5,\ x^{-5} = \frac{1}{x^{5}}.$

- Both $-9x^{-5} + c$ and $-\frac{9}{x^5} + c$ are correct answers.

4

We have

$$\int \frac{3}{x}\,dx = \int 3x^{-1}\,dx = 3\ln x + c$$

Check the answer: Let $y = 3\ln x + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}(3\ln x + c) = \frac{d}{dx}(3\ln x) + \frac{d}{dx}(c) = \frac{3}{x} + 0 = \frac{3}{x}$$

Explanation:

- Recall that when the exponent n of x is $n = -1$, we have $\int x^{-1}\,dx = \ln x + c$.

5

$$\int 5x^{4/7}\,dx = \frac{5x^{4/7+1}}{\frac{4}{7}+1} + c = \frac{5x^{11/7}}{11/7} + c = \frac{35x^{11/7}}{11} + c$$

Check the answer: Let $y = \frac{35x^{11/7}}{11} + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{35x^{11/7}}{11} + c\right) = \left(\frac{35}{11}\right)\left(\frac{11}{7}\right)x^{11/7-1} + 0 = 5x^{4/7}$$

Explanation:

- Recall that fractions can be added or subtracted only when they have common denominator: $\frac{4}{7} + 1 = \frac{4}{7} + \frac{7}{7} = \frac{4+7}{7} = \frac{11}{7}$ and $\frac{11}{7} - 1 = \frac{11}{7} - \frac{7}{7} = \frac{11-7}{7} = \frac{4}{7}$.
- $\frac{5}{11/7} = 5 \times \frac{7}{11} = \frac{35}{11}$.

Set B

6

$$\int 6\,dt = \int 6t^0\,dt = \frac{6t^{0+1}}{0+1} + c = 6\frac{t^1}{1} + c = 6t + c$$

Check the answer: Let $y = 6t + c,$ then

$$\frac{dy}{dt} = \frac{d}{dt}(6t + c) = \frac{d}{dt}(6t) + \frac{d}{dt}(c) = 6 + 0 = 6$$

Explanation:

- Note that $1 = t^0$ and $t^1 = t$.

We have

$$\int 4x^{-3}\,dx = \frac{4x^{-3+1}}{-3+1} + c = \frac{4x^{-2}}{-2} + c = -2x^{-2} + c = -\frac{2}{x^2} + c$$

Check the answer: Let $y = -2x^{-2} + c$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(-2x^{-2} + c\right) = \frac{d}{dx}\left(-2x^{-2}\right) + \frac{d}{dx}(c)$$

$$= (-2)(-2)x^{-2-1} + 0 = 4x^{-3}$$

Explanation:

- Note that $-3 + 1 = -2,\ x^{-2} = \frac{1}{x^2}.$
- Both $-2x^{-2} + c$ and $-\frac{2}{x^2} + c$ are correct answers.

8

We have

$$\int 2x\,dx = \int 2x^1\,dx = \frac{2x^{1+1}}{1+1} + c = \frac{2x^2}{2} + c = x^2 + c$$

Check the answer: Let $y = x^2 + c$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + c\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(c) = 2x + 0 = 2x$$

Explanation:

- Note that $x = x^1$.

9

We have

$$\int \frac{3}{x^{3/4}}\,dx = \int 3x^{-3/4}\,dx = \frac{3x^{-3/4+1}}{-\frac{3}{4}+1} + c = \frac{3x^{1/4}}{1/4} + c = 12x^{1/4} + c$$

Check the answer: Let $y = 12x^{1/4} + c$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(12x^{1/4} + c\right) = (12)\left(\frac{1}{4}\right)x^{1/4-1} + 0 = 3x^{-3/4} = \frac{3}{x^{3/4}}$$

Explanation:

- Since $\frac{1}{x^n} = x^{-n}$, we have $\frac{1}{x^{3/4}} = x^{-3/4}$.
- $-\frac{3}{4} + 1 = -\frac{3}{4} + \frac{4}{4} = \frac{-3+4}{4} = \frac{1}{4}$ (fractions can be added or subtracted only when they have common denominator)
- $\frac{3}{1/4} = 3 \times \frac{4}{1} = \frac{12}{1} = 12.$

Set C

10 We have

$$\int(5x^5 + 10x^9)\,dx = \int 5x^5\,dx + \int 10x^9\,dx = \frac{5x^{5+1}}{5+1} + \frac{10x^{9+1}}{9+1} + c$$

$$= \frac{5x^6}{6} + \frac{10x^{10}}{10} + c = \frac{5x^6}{6} + x^{10} + c$$

Check the answer: Let $y = \frac{5x^6}{6} + x^{10} + c$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{5x^6}{6} + x^{10} + c\right) = \frac{d}{dx}\left(\frac{5x^6}{6}\right) + \frac{d}{dx}(x^{10}) + \frac{d}{dx}(c)$$

$$= \left(\frac{5}{6}\right)(6)x^5 + 10x^9 + 0 = 5x^5 + 10x^9$$

Explanation:

- Note that we have used the addition rule for integration to convert the integral of a polynomial to a sum of integrals of individual terms.

11 We have

$$\int\left(3x^3 + 7 + \frac{4}{x}\right)dx = \int 3x^3dx + \int 7x^0dx + \int 4x^{-1}dx$$

$$= \frac{3x^{3+1}}{3+1} + \frac{7x^{0+1}}{0+1} + 4\ln x + c = \frac{3x^4}{4} + \frac{7x}{1} + 4\ln x + c = \frac{3x^4}{4} + 7x + 4\ln x + c$$

Check the answer: Let $y = \frac{3x^4}{4} + 7x + 4\ln x + c$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x^4}{4} + 7x + 4\ln x + c\right) = \frac{d}{dx}\left(\frac{3x^4}{4}\right) + \frac{d}{dx}(7x) + \frac{d}{dx}(4\ln x) + \frac{d}{dx}(c)$$

$$= \left(\frac{3}{4}\right)(4)x^{4-1} + (7)(1)x^{1-1} + \frac{4}{x} + 0 = 3x^3 + 7x^0 + \frac{4}{x} = 3x^3 + 7 + \frac{4}{x}$$

Explanation:

- Recall that when the exponent n of x is $n = -1$, we have $\int x^{-1}\,dx = \ln x + c$.

12

We have

$$\int (3x^{2/3} + 6x^{1/3})dx = \int 3x^{2/3}\,dx + \int 6x^{1/3}\,dx$$

$$= \frac{3x^{2/3+1}}{2/3+1} + \frac{6x^{1/3+1}}{1/3+1} + c = \frac{3x^{5/3}}{5/3} + \frac{6x^{4/3}}{4/3} + c = \frac{9}{5}x^{5/3} + \frac{9}{2}x^{4/3} + c$$

Check the answer: Let $y = \frac{9}{5}x^{5/3} + \frac{9}{2}x^{4/3} + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{9}{5}x^{5/3} + \frac{9}{2}x^{4/3} + c\right) = \frac{d}{dx}\left(\frac{9}{5}x^{5/3}\right) + \frac{d}{dx}\left(\frac{9}{2}x^{4/3}\right) + \frac{d}{dx}(c)$$

$$= \left(\frac{9}{5}\right)\left(\frac{5}{3}\right)x^{(5/3-1)} + \left(\frac{9}{2}\right)\left(\frac{4}{3}\right)x^{(4/3-1)} + 0 = 3x^{2/3} + 6x^{1/3}$$

Explanation:

- $\frac{2}{3} + 1 = \frac{2}{3} + \frac{3}{3} = \frac{2+3}{3} = \frac{5}{3}$ and $\frac{1}{3} + 1 = \frac{1}{3} + \frac{3}{3} = \frac{1+3}{3} = \frac{4}{3}$.
- $\frac{3}{\frac{5}{3}} = 3 \times \frac{3}{5} = \frac{9}{5}$ and $\frac{6}{\frac{4}{3}} = 6 \times \frac{3}{4} = \frac{18}{4} = \frac{2 \times 9}{2 \times 2} = \frac{9}{2}$.

13

We have

$$\int\left(\frac{3x^3 + 5x^2 + 8}{x^2}\right)dx = \int\left(\frac{3x^3}{x^2} + \frac{5x^2}{x^2} + \frac{8}{x^2}\right)dx$$

$$= \int\left(3x + 5 + \frac{8}{x^2}\right)dx = \int 3x^1\,dx + \int 5x^0\,dx + \int 8x^{-2}\,dx$$

$$= \frac{3x^{1+1}}{1+1} + \frac{5x^{0+1}}{0+1} + \frac{8x^{-2+1}}{-2+1} + c = \frac{3x^2}{2} + \frac{5x^1}{1} + \frac{8x^{-1}}{-1} + c = \frac{3}{2}x^2 + 5x - \frac{8}{x} + c$$

Check the answer: Let $y = \frac{3}{2}x^2 + 5x - \frac{8}{x} + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3}{2}x^2 + 5x - \frac{8}{x} + c\right) = \frac{d}{dx}\left(\frac{3}{2}x^2\right) + \frac{d}{dx}(5x) + \frac{d}{dx}(-8x^{-1}) + \frac{d}{dx}(c)$$

$$= \left(\frac{3}{2}\right)(2)x^{(2-1)} + 5 - (8)(-1)x^{(-1-1)} + 0 = 3x + 5 + 8x^{-2} = 3x + 5 + \frac{8}{x^2}$$

$$= \frac{3x^3 + 5x^2 + 8}{x^2}$$

Explanation:

- Note that $x = x^1$ and $1 = x^0$.
- $3x + 5 + \frac{8}{x^2} = \frac{3x}{1} \times \frac{x^2}{x^2} + \frac{5}{1} \times \frac{x^2}{x^2} + \frac{8}{x^2} = \frac{3x^3}{x^2} + \frac{5x^3}{x^2} + \frac{8}{x^2} = \frac{3x^3 + 5x^2 + 8}{x^2}$ (fractions can be added or subtracted only when they have common denominator).

15 DEFINITE INTEGRALS

In the previous chapter, we have learnt that the general antiderivative with arbitrary constant c is known as the indefinite integral of $f(x)$ with respect to x.

In the present chapter, we will study definite integral. A definite integral is an integral with limits. It is denoted as $\int_{x=a}^{b} f(x)\, dx$ where a is called the lower limit of the integral and b is called the upper limit of the integral.

The basic theorem of calculus allows definite integrals to be computed in terms of antiderivatives. According to this theorem, if $f(x)$ has an antiderivative $F(x)$, then the value of $\int_{x=a}^{b} f(x)\, dx$ is the difference between the values of $F(x)$ at the upper limit and the lower limit:

$$\int_{x=a}^{b} f(x)\, dx = F(b) - F(a)$$

To make use of this theorem, follow these steps:

1. Find the antiderivative $F(x)$ of $f(x)$.
2. Calculate the values of antiderivative at the upper limit $[= F(b)]$ and at the lower limit $[= F(a)]$.
3. Evaluate $F(b) - F(a)$.
4. Then $F(b) - F(a)$ is the value of $\int_{x=a}^{b} f(x)\, dx$.

Note that there is no need to include the arbitrary constant c while writing the antiderivative in step 1. This is because c gets cancelled when we evaluate the difference $F(b) - F(a)$ in step 2.

We will use the notation $[F(x)]_a^b$ to denote $F(b) - F(a)$. Using this notation, we can write

$$\int_{x=a}^{b} f(x)\, dx = [F(x)]_a^b$$

Now, study carefully the following examples to learn the use of the above formula for evaluating the definite integrals and then attempt the exercises for this chapter.

Example 1: Evaluate the following integral

$$\int_{x=2}^{4} 7x^3\, dx$$

Solution: We have

$$\int_{x=2}^{4} 7x^3\, dx$$

$$= \left[\frac{7x^{3+1}}{3+1}\right]_{x=2}^{4} = \left[\frac{7x^4}{4}\right]_{x=2}^{4} = \frac{7(4^4)}{4} - \frac{7(2^4)}{4} = \frac{7}{4}(256 - 16) = 7(60)$$

$$= 420$$

Check: We will check that $(7x^4/4)$ is the anti-derivative of $(7x^3)$.

$$\frac{d}{dx}\left(\frac{7x^4}{4}\right) = \frac{7}{4}\left(4x^{4-1}\right) = \frac{7}{4}\left(4x^3\right) = 7x^3$$

Example 2: Evaluate the following integral

$$\int_{t=-1}^{2} \left(3t^3 + 5\right) dt$$

Solution: Here, the integration variable is *t*. We, therefore, need to replace *x* with *t* in various formulas.

$$\int_{t=-1}^{2} \left(3t^3 + 5\right) dt = \left[\frac{3t^{3+1}}{3+1} + 5\left(\frac{t^{0+1}}{0+1}\right)\right]_{t=-1}^{2} = \left[\frac{3t^4}{4} + \frac{5t}{1}\right]_{t=-1}^{2}$$

$$= \left(\frac{3(2)^4}{4} + \frac{5(2)}{1}\right) - \left(\frac{3(-1)^4}{4} + \frac{5(-1)}{1}\right) = \left(\frac{3(16)}{4} + \frac{5(2)}{1}\right) - \left(\frac{3(1)}{4} + \frac{5(-1)}{1}\right)$$

$$= \left(12 + \frac{10}{1}\right) - \left(\frac{3}{4} - \frac{5}{1}\right) = (12 + 10) - \left(\frac{3}{4} - \frac{20}{4}\right) = 22 - \left(-\frac{17}{4}\right) = 22 + \frac{17}{4}$$

$$= \frac{88}{4} + \frac{17}{4} = \frac{88 + 17}{4} = \frac{105}{4} \approx 26.25$$

Check: We will check that $\left(\frac{3t^4}{4} + \frac{4t^3}{3} + \frac{5t}{1}\right)$ is the anti-derivative of $\left(3t^3 + 4t^2 + 5\right)$.

$$\frac{d}{dt}\left(\frac{3t^4}{4} + \frac{4t^3}{3} + \frac{5t}{1}\right) = \frac{d}{dt}\left(\frac{3t^4}{4}\right) + \frac{d}{dt}\left(\frac{4t^3}{3}\right) + \frac{d}{dt}\left(\frac{5t}{1}\right) = \frac{3}{4}\left(4t^3\right) + \frac{4}{3}\left(3t^2\right) + 5(1)$$

$$= 3t^3 + 4t^2 + 5$$

Chapter 15 Exercises

Set A

Evaluate the following integrals.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\int_{x=1}^{2} \frac{1}{x^5}\, dx =$$

2

$$\int_{x=1}^{3} \frac{4}{x}\, dx =$$

3

$$\int_{x=1}^{8} 4x^{1/3}\, dx =$$

4

$$\int_{x=4}^{9} \sqrt{x}\, dx =$$

Set B

5

$$\int_{x=-2}^{2} \left(3x^2 + 5x\right) dx \ =$$

6

$$\int_{x=1}^{4} \left(3\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx =$$

7

$$\int_{x=2}^{4} \left(\frac{3}{x} + 2\right) dx =$$

$$\int_{x=1}^{9} x\sqrt{x}\, dx =$$

Chapter 15 Solutions

Set A

1 We have

$$\int_{x=1}^{2} \frac{1}{x^5}\,dx = \int_{x=1}^{2} x^{-5}\,dx = \left[\frac{x^{-5+1}}{-5+1}\right]_{x=1}^{2} = \left[\frac{x^{-4}}{-4}\right]_{x=1}^{2} = \left(-\frac{1}{4}\right)\left[x^{-4}\right]_{x=1}^{2}$$

$$= \left(-\frac{1}{4}\right)\left[\frac{1}{x^4}\right]_{x=1}^{2} = \left(-\frac{1}{4}\right)\left(\frac{1}{2^4} - \frac{1}{1^4}\right) = \left(-\frac{1}{4}\right)\left(\frac{1}{16} - \frac{1}{1}\right) = \left(-\frac{1}{4}\right)\left(\frac{1}{16} - \frac{16}{16}\right)$$

$$= \left(-\frac{1}{4}\right)\left(\frac{1-16}{16}\right) = \left(-\frac{1}{4}\right)\left(-\frac{15}{16}\right) = \frac{15}{64} \approx 0.234$$

Check: We will check that $\left(\frac{x^{-4}}{-4}\right)$ is the anti-derivative of $(1/x^5)$.

$$\frac{d}{dx}\left(\frac{x^{-4}}{-4}\right) = \frac{d}{dx}\left(-\frac{x^{-4}}{4}\right) = \left(-\frac{1}{4}\right)(-4)x^{-5} = x^{-5} = \frac{1}{x^5}$$

2 We have

$$\int_{x=1}^{3} \frac{4}{x}\,dx = 4\int_{x=1}^{3} x^{-1}\,dx = 4[\ln x]_{x=1}^{3} = 4(\ln 3 - \ln 1) = 4(\ln 3 - 0) = 4\ln 3$$

$$\approx 4.394.$$

Check: We will check that $(4\ln x)$ is the anti-derivative of $(4/x)$.

$$\frac{d}{dx}(4\ln x) = \frac{4}{x}$$

Explanation:

Recall that when the exponent n of x is $n = -1$, we have $\int x^{-1}\,dx = \ln x + c$. We drop the arbitrary constant c because it gets cancelled when we subtract after evaluating the antiderivative at upper and lower limits. Note also that $\ln 1 = 0$. We have used calculator to estimate that $4\ln 3 \approx 4.394.$

3

We have

$$\int_{x=1}^{8} 4x^{1/3}\,dx = \left[\frac{4x^{1/3+1}}{\frac{1}{3}+1}\right]_{x=1}^{8} = \left[\frac{4x^{4/3}}{\frac{4}{3}}\right]_{x=1}^{8} = \left[3x^{4/3}\right]_{x=1}^{8}$$

$$= 3(8)^{4/3} - 3(1)^{4/3} = 3(2)^4 - 3(1) = 3(16) - 3(1) = 48 - 3 = 45$$

Check: We will check that $(3x^{4/3})$ is the anti-derivative of $(4x^{1/3})$.

$$\frac{d}{dx}(3x^{4/3}) = (3)\left(\frac{4}{3}\right)x^{1/3} = 4x^{1/3}$$

Explanation: $\frac{1}{3} + 1 = \frac{1}{3} + \frac{3}{3} = \frac{1+3}{3} = \frac{4}{3}$, $\frac{4}{\frac{4}{3}} = 4 \times \frac{3}{4} = \frac{12}{4} = 3$

4

We have

$$\int_{x=4}^{9} \sqrt{x}\,dx$$

$$= \int_{x=4}^{9} x^{1/2}\,dx = \left[\frac{x^{1/2+1}}{1/2+1}\right]_{x=4}^{9} = \left[\frac{x^{3/2}}{3/2}\right]_{x=4}^{9} = \left[\frac{2x^{3/2}}{3}\right]_{x=4}^{9} = \frac{2(9)^{3/2}}{3} - \frac{2(4)^{3/2}}{3}$$

$$= \frac{2(27)}{3} - \frac{2(8)}{3} = \frac{54-16}{3} = \frac{38}{3} \approx 12.667$$

Check: We will check that $\left(\frac{2x^{3/2}}{3}\right)$ is the anti-derivative of $(\sqrt{x})$.

$$\frac{d}{dx}\left(\frac{2x^{3/2}}{3}\right) = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{1/2} = \sqrt{x}$$

Explanation:

- $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$ (fractions can be added or subtracted only when they have common denominator)
- Recalling that $x^{m/n} = (x^{1/n})^m$, we have $(9)^{3/2} = (9^{1/2})^3 = 3^3 = 27$ and $(4)^{3/2}$ $(4)^{3/2} = (4^{1/2})^3 = 2^3 = 8.$

5 Set B

We have

$$\int_{x=-2}^{2} (3x^2 + 5x)\, dx = \left[\frac{3x^{2+1}}{2+1} + \frac{5x^{1+1}}{1+1}\right]_{x=-2}^{2} = \left[\frac{3x^3}{3} + \frac{5x^2}{2}\right]_{x=-2}^{2}$$

$$= \left[x^3 + \frac{5x^2}{2}\right]_{x=-2}^{2} = \left((2)^3 + \frac{5(2)^2}{2}\right) - \left((-2)^3 + \frac{5(-2)^2}{2}\right)$$

$$= (8 + 10) - (-8 + 10) = (18) - (2) = 16$$

Check: We will check that $\left(x^3 + \frac{5x^2}{2}\right)$ is the anti-derivative of $(3x^2 + 5x)$.

$$\frac{d}{dx}\left(x^3 + \frac{5x^2}{2}\right) = \frac{d}{dx}(x^3) + \frac{d}{dx}\left(\frac{5x^2}{2}\right) = 3x^2 + \left(\frac{5}{2}\right)(2)x = 3x^2 + 5x$$

6

We have

$$\int_{x=1}^{4} \left(3\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx = \int_{x=1}^{4} \left(3x^{1/2} + \frac{x^{-1/2}}{2}\right) dx = \left[\frac{3x^{1/2+1}}{\frac{1}{2}+1} + \frac{x^{(-1/2+1)}}{2\left(-\frac{1}{2}+1\right)}\right]_{x=1}^{4}$$

$$= \left[\frac{3x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{2\left(\frac{1}{2}\right)}\right]_{x=1}^{4} = \left[2x^{3/2} + x^{1/2}\right]_{x=1}^{4}$$

$$= \left(2(4)^{3/2} + (4)^{1/2}\right) - \left(2(1)^{3/2} + (1)^{1/2}\right)$$

$$= (16 + 2) - (2 + 1) = 18 - 3 = 15$$

Check: We will check that $(2x^{3/2} + x^{1/2})$ is the anti-derivative of $\left(3\sqrt{x} + \frac{1}{2\sqrt{x}}\right)$.

$$\frac{d}{dx}(2x^{3/2} + x^{1/2}) = (2)\left(\frac{3}{2}x^{1/2}\right) + \frac{1}{2}x^{-1/2} = 3x^{1/2} + \frac{1}{2}x^{-1/2} = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$$

Explanation:

- $\sqrt{x} = x^{1/2}$. Since $\frac{1}{x^n} = x^{-n}$, we have $\frac{1}{x^{1/2}} = x^{-1/2}$.
- $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$, $-\frac{1}{2} + 1 = -\frac{1}{2} + \frac{2}{2} = \frac{-1+2}{2} = \frac{1}{2}$ and $3 \times \frac{2}{3} = 2$.

7 We have

$$\int_{x=2}^{4}\left(\frac{3}{x}+2\right)dx$$

$$= \int_{x=2}^{4}\left(3x^{-1}+2x^{0}\right)dx = \left[3\ln x+\frac{2x}{1}\right]_{x=2}^{4} = [3\ln x+2x]_{x=2}^{4}$$

$$= [3\ln 4+2(4)] - [3\ln 2+2(2)] = 3\ln 4 - 3\ln 2 + 8 - 4$$

$$= 3(\ln 4 - \ln 2) + (8-4) = 3\ln\left(\frac{4}{2}\right) + 4 = 3\ln 2 + 4 \approx 6.079$$

Check: We will check that $(3\ln x+2x)$ is the anti-derivative of $\left(\frac{3}{x}+2\right)$.

$$\frac{d}{dx}(3\ln x+2x) = \frac{d}{dx}(3\ln x) + \frac{d}{dx}(2x) = \frac{3}{x}+2$$

Explanation:

- Recall that when the exponent n of x is n =- 1, we have $\int x^{-1}\,dx = \ln x + c$. We drop the arbitrary constant c because it gets cancelled when we subtract after evaluating the antiderivative at upper and lower limits.
- Note that $2 = 2x^{0}$ and $x^{1} = x$.
- Recall the property: $\ln x - \ln y = \ln\frac{x}{y}$, which gives $\ln 4 - \ln 2 = \ln\left(\frac{4}{2}\right) = \ln 2.$

8

$$\int_{x=1}^{9} x\sqrt{x}\,dx = \int_{x=1}^{9} xx^{1/2}\,dx = \int_{x=1}^{9} x^{3/2}\,dx = \left[\frac{x^{3/2+1}}{\frac{3}{2}+1}\right]_{x=1}^{9} = \left[\frac{x^{5/2}}{\frac{5}{2}}\right]_{x=1}^{9}$$

$$= \left[\frac{2x^{5/2}}{5}\right]_{x=1}^{9} = \left(\frac{2}{5}\right)[x^{5/2}]_{x=1}^{9} = \left(\frac{2}{5}\right)\left[(9)^{5/2} - (1)^{5/2}\right] = \left(\frac{2}{5}\right)(243-1) = \left(\frac{2}{5}\right)(242) = \frac{484}{5} = 96.8$$

Check: We will check that $\left(\frac{2x^{5/2}}{5}\right)$ is the anti-derivative of $(x\sqrt{x})$.

$$\frac{d}{dx}\left(\frac{2x^{5/2}}{5}\right) = \frac{2}{5}\left(\frac{5}{2}x^{3/2}\right) = x^{3/2} = x\sqrt{x}$$

Explanation:

- Since $x^{m}x^{n} = x^{m+n}$, we have $x\sqrt{x} = xx^{1/2} = x^{1+1/2} = x^{1+\frac{1}{2}} = x^{3/2}$, $1+\frac{1}{2} = \frac{2}{2}+\frac{1}{2} = \frac{2+1}{2} = \frac{3}{2}$.

- Note that $x^{m/n} = (x^{1/n})^m$ gives $(9)^{5/2} = (9^{1/2})^5 = 3^5 = 243$.

16 INTEGRALS OF TRIGONOMETRIC FUNCTIONS

We start by giving below the derivatives of basic trigonometric functions.

$$\int \sin\theta d\theta = -\cos\theta + c \qquad \int \cos\theta d\theta = \sin\theta + c$$

$$\int \tan\theta d\theta = \ln|\sec\theta| + c \qquad \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$\int \cot\theta d\theta = \ln|\sin\theta| + c \qquad \int \csc\theta d\theta = -\ln|\csc\theta + \cot\theta| + c$$

Before proceeding further, let us recall the following important relations and identities:

$$\sin\theta = \frac{1}{\csc\theta} \text{ or } \csc\theta = \frac{1}{\sin\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \text{ or } \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Example 1: Evaluate the following integral

$$\int_{\theta=0}^{\pi/4} \sin\theta d\theta$$

Solution: We have

$$\int_{\theta=0}^{\pi/4} \sin\theta d\theta = [-\cos\theta]_{\theta=0}^{\pi/4} = -\cos\frac{\pi}{4} - (-\cos 0) = -\frac{1}{\sqrt{2}} - (-1) = -\frac{1}{\sqrt{2}} + 1$$

$$= -\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} = \frac{-1+\sqrt{2}}{\sqrt{2}} = \frac{(-1+\sqrt{2})\sqrt{2}}{\sqrt{2}\quad\sqrt{2}} = \frac{-\sqrt{2}+2}{2} \approx 0.293$$

$$\text{Check: } \frac{dy}{d\theta} = \frac{d}{d\theta}(-\cos\theta) = -\frac{d}{d\theta}(\cos\theta) = -(-\sin\theta) = \sin\theta$$

Chapter 16 Exercises

Evaluate the following integrals.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\int_{\theta = 0}^{\pi/3} \cos\theta \, d\theta =$$

2

$$\int_{\theta = \pi/4}^{\pi/3} \cot\theta \, d\theta =$$

3

$$\int_{\theta = 0}^{\pi/4} \tan\theta \, d\theta =$$

4

$$\int_{\theta = 0}^{\pi/3} (\theta + \sin\theta) \, d\theta =$$

$$\int_{\theta = \pi/4}^{\pi/3} \csc\theta \, d\theta =$$

Chapter 16 Solutions

1

$$\int_{\theta=0}^{\pi/3} \cos\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/3} \cos\theta \, d\theta = [\sin\theta]_{\theta=0}^{\pi/3} = \left[\sin\frac{\pi}{3} - \sin 0\right] = \left[\frac{\sqrt{3}}{2} - 0\right] = \frac{\sqrt{3}}{2} \approx 0.866$$

Explanation:

- $\frac{\pi}{3}$ radians $= \frac{180^0}{3} = 60^0$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin 0 = 0$.

2

$$\int_{\theta=\pi/4}^{\pi/3} \cot\theta \, d\theta = \int_{\theta=\pi/4}^{\pi/3} \cot\theta \, d\theta = [\ln|\sin\theta|]_{\theta=\pi/4}^{\pi/3} = \ln\left|\sin\frac{\pi}{3}\right| - \ln\left|\sin\frac{\pi}{4}\right|$$

$$= \ln\frac{\sqrt{3}}{2} - \ln\frac{1}{\sqrt{2}} = \ln\left(\frac{\sqrt{3}}{\sqrt{2}}\right) = \ln\left(\sqrt{\frac{3}{2}}\right) = \ln\left(\frac{3}{2}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{3}{2}\right) = \frac{1}{2}\ln(1.5) \approx 0.203$$

Explanation:

- $\frac{\pi}{3}$ radians $= \frac{180^0}{3} = 60^0$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{4}$ radians $= \frac{180^0}{4} = 45^0$, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.
- Since $\ln x - \ln y = \ln\frac{x}{y}$, $\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{\sqrt{2}} = \ln\left(\frac{\sqrt{3}}{2} \div \frac{1}{\sqrt{2}}\right) = \ln\left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{1}\right) = \ln\left(\frac{\sqrt{3}}{\sqrt{2}}\right)$.
- $\ln x^n = n\ln x$, which gives $\ln\left(\frac{3}{2}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\frac{3}{2} = \frac{1}{2}\ln(1.5)$.
- We have used calculator to estimate that $\frac{1}{2}\ln(1.5) \approx 0.203$.

3

$$\int_{\theta=0}^{\pi/4} \tan\theta \, d\theta = [\ln|\sec\theta|]_{\theta=0}^{\pi/4} = \ln\left(\sec\frac{\pi}{4}\right) - \ln(\sec 0) = \ln(\sqrt{2}) - \ln(1)$$

$$= \ln\sqrt{2} - 0 = \ln\sqrt{2} = \ln 2^{1/2} = \frac{1}{2}\ln 2 \approx 0.347$$

Explanation:

- $\frac{\pi}{4}$ radians $= \frac{180^0}{4} = 45^0$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$, $\sec 0 = \frac{1}{\cos 0} = 1$.

- $\ln(1) = 0$, $\ln 2^{1/2} = \frac{1}{2}\ln 2$, since $\ln x^n = n\ln x$.
- We have used calculator to estimate the numerical value.

4

$$\int_{\theta=0}^{\pi/3} (\theta + \sin\theta)\, d\theta = \int_{\theta=0}^{\pi/3} \theta\, d\theta + \int_{\theta=0}^{\pi/3} \sin\theta\, d\theta$$

$$= \left[\frac{\theta^2}{2}\right]_{\theta=0}^{\pi/3} + [-\cos\theta]_{\theta=0}^{\pi/3}$$

$$= \left[\frac{\left(\frac{\pi}{3}\right)^2}{2} - \frac{0^2}{2}\right] + \left[-\cos\frac{\pi}{3} - (-\cos 0)\right] = \frac{\left(\frac{\pi^2}{9}\right)}{2} - 0 + \left[-\left(\frac{1}{2}\right) - (-1)\right]$$

$$= \frac{\pi^2}{18} + \left(-\frac{1}{2} + 1\right) = \frac{\pi^2}{18} + \frac{1}{2} \approx 1.048$$

Check: We will check that $\left(\frac{\theta^2}{2} - \cos\theta\right)$ is the anti-derivative of $(\theta + \sin\theta)$.

$$\frac{d}{dx}\left(\frac{\theta^2}{2} - \cos\theta\right) = \frac{1}{2}(2\theta) - (-\sin\theta) = \theta + \sin\theta$$

Explanation:

- We have used the addition rule for integration to get $\int(\theta + \sin\theta)\, d\theta = \int\theta\, d\theta + \int\sin\theta\, d\theta$.
- $\frac{\pi}{3}$ radians $= \frac{180^0}{3} = 60^0$, $\cos\frac{\pi}{3} = \frac{1}{2}$.
- $\cos 0 = 1$.
- Note that $\frac{\left(\frac{\pi^2}{9}\right)}{2} = \frac{\pi^2}{9} \div \frac{2}{1} = \frac{\pi^2}{9} \times \frac{1}{2} = \frac{\pi^2}{18}$.
- $-(-1) = +1$.
- $-\frac{1}{2} + 1 = -\frac{1}{2} + \frac{2}{2} = \frac{-1+2}{2} = \frac{1}{2}$.
- We have used calculator to estimate that $\frac{\pi^2}{18} + \frac{1}{2} \approx 1.048$.

[redacted]

$$\int_{\theta = \pi/4}^{\pi/3} \csc\theta \, d\theta = [-\ln|\csc\theta + \cot\theta|]_{\theta = \pi/4}^{\pi/3}$$

$$= -\ln\left|\csc\frac{\pi}{3} + \cot\frac{\pi}{3}\right| - \left[-\ln\left|\csc\frac{\pi}{4} + \cot\frac{\pi}{4}\right|\right] = -\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - (-\ln|\sqrt{2} + 1|)$$

$$= -\ln\left|\frac{2+1}{\sqrt{3}}\right| + \ln|\sqrt{2}+1| = -\ln\left|\frac{3}{\sqrt{3}}\right| + \ln|\sqrt{2}+1| = -\ln|\sqrt{3}| + \ln|\sqrt{2}+1|$$

$$= \ln|\sqrt{2}+1| - \ln|\sqrt{3}| = \ln\left(\frac{\sqrt{2}+1}{\sqrt{3}}\right) \approx 0.332$$

Explanation:

- $\frac{\pi}{3}$ radians $= \frac{180^0}{3} = 60^0$, $\frac{\pi}{4}$ radians $= \frac{180^0}{4} = 45^0$.
- $\csc\frac{\pi}{3} = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\sin(60^0)} = \frac{1}{\frac{\sqrt{3}}{2}} = 1 \times \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$, $\cot\frac{\pi}{3} = \frac{1}{\tan\frac{\pi}{3}} = \frac{1}{\sqrt{3}}$, $\csc\frac{\pi}{4} = \frac{1}{\sin\left(\frac{\pi}{4}\right)}$ $= \frac{1}{\frac{1}{\sqrt{2}}} = 1 \times \frac{\sqrt{2}}{1} = \sqrt{2}$, $\cot\left(\frac{\pi}{4}\right) = \frac{1}{\tan\left(\frac{\pi}{4}\right)} = \frac{1}{1} = 1$.
- $-\ln|\sqrt{3}| + \ln|\sqrt{2}+1| = \ln|\sqrt{2}+1| - \ln|\sqrt{3}| = \ln\left|\frac{\sqrt{2}+1}{\sqrt{3}}\right|$, since $\ln x - \ln y = \ln\frac{x}{y}$.
- We have used calculator to estimate that $\ln\left(\frac{\sqrt{2}+1}{\sqrt{3}}\right) \approx 0.332$.

17 INTEGRALS OF EXPONENTIAL, LOGARITHMIC AND HYPERBOLIC FUNCTIONS

The integrals of basic exponential, logarithmic and power functions are given below.

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int b^{x}\,dx = \frac{b^{x}}{\ln b} + c$$

The integrals of hyperbolic functions are:

$$\int \sinh x\,dx = \cosh x + c \quad , \qquad \int \cosh x\,dx = \sinh x$$

$$\int \tanh x\,dx = \ln \cosh x + c$$

Before proceeding further, let us recall the following important relations and identities:

$$e = 2.7118281828....,\qquad e^{x+y} = e^{x}e^{y},\qquad e^{-x} = \frac{1}{e^{x}},$$

$$e^{x-y} = e^{x}e^{-y} = \frac{e^{x}}{e^{y}},\qquad (e^{x})^{a} = e^{ax},\qquad e^{0} = 1$$

$$\ln(xy) = \ln x + \ln y,\qquad \ln\frac{x}{y} = \ln x - \ln y,\qquad \ln x^{n} = n\ln x$$

$$\ln(e^{x}) = x,\qquad \ln e = 1,\qquad \ln 1 = 0$$

$$\log_{b} x = \frac{\ln x}{\ln b}\qquad e^{\ln x} = x,\qquad \ln\frac{1}{x} = \ln x^{-1} = -\ln x$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2},\qquad \cosh x = \frac{e^{x} + e^{-x}}{2},\quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1, \qquad \sinh(-x) = -\sinh x, \qquad \cosh(-x) = \cosh x$$

$$\sinh 0 = 0, \qquad \cosh 0 = 1, \qquad \tanh 0 = 0$$

Example 1: Evaluate the following integral

$$\int_{x=0}^{1} (e^x + e^{-x})\, dx$$

Solution: We have

$$\int_{x=0}^{1} (e^x + e^{-x})\, dx = \left[e^x + \frac{e^{-x}}{-1}\right]_{x=0}^{1} = \left[e^x + (-e^{-x})\right]_{x=0}^{1} = \left[e^x - e^{-x}\right]_{x=0}^{1}$$

$$= (e^1 - e^{-1}) - (e^0 - e^{-0}) = \left(e - \frac{1}{e}\right) - (1 - 1) = \left(e - \frac{1}{e}\right) - 0 = e - \frac{1}{e} \approx 2.350$$

Check: We will check that $(e^x - e^{-x})$ is the anti-derivative of $(e^x + e^{-x})$.

Let $y = e^x - e^{-x}$, then

$$\frac{dy}{dx} = \frac{d}{dx}(e^x - e^{-x}) = \frac{d}{dx}(e^x) + \frac{d}{dx}(-e^{-x}) = \frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})$$

$$= e^x - (-e^{-x}) = e^x + e^{-x}$$

Example 2: Evaluate the following integral

$$\int_{x=1}^{3} 3^x\, dx$$

Solution: We have

$$\int_{x=1}^{3} 3^x\, dx = \left[\frac{3^x}{\ln 3}\right]_{x=1}^{3} = \frac{3^3}{\ln 3} - \frac{3^1}{\ln 3} = \frac{3^3 - 3^1}{\ln 3} = \frac{27 - 3}{\ln 3} = \frac{24}{\ln 3} \approx 21.846$$

Check: We will check that $\left(\frac{3^x}{\ln 3}\right)$ is the anti-derivative of (3^x).

Let $y = \frac{3^x}{\ln 3}$, then

$$\frac{dy}{dx} = \frac{d}{dx}\frac{3^x}{\ln 3} = \frac{1}{\ln 3}\frac{d}{dx}(3^x) = \frac{1}{\ln 3}(3^x \ln 3) = \frac{\ln 3}{\ln 3}(3^x) = 3^x$$

Chapter 17 Exercises

Set A

Evaluate the following integrals.

You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\int_{x=1}^{\infty} e^{-x}\, dx =$$

2

$$\int_{x=-1}^{1} \cosh x\, dx =$$

Set B

$$\int_{x=1}^{e} (\ln x + x)\, dx =$$

$$\int_{x=1}^{2} 4^{x}\, dx =$$

5

$$\int_{x=0}^{1} (x + \sinh x)\, dx =$$

Chapter 17 Solutions

Set A

1

$$\int_{x=1}^{\infty} e^{-x}\,dx = ?$$

This is an example of improper integral wherein one of the limits of integration (the upper limit in this case) grows to infinity. To evaluate this integral, we assume the upper limit to be $x = k$ and study the limit as k approaches infinity:

$$\int_{x=1}^{\infty} e^{-x}\,dx = \lim_{k\to\infty}\int_{x=1}^{k} e^{-x}\,dx = \lim_{k\to\infty}\left[\frac{e^{-x}}{-1}\right]_{x=1}^{k} = \lim_{k\to\infty}\left[-\frac{1}{e^{x}}\right]_{x=1}^{k}$$

$$= \lim_{k\to\infty}\left[-\left(\frac{1}{e^{k}}-\frac{1}{e^{1}}\right)\right] = \lim_{k\to\infty}\ -\left(\frac{1}{e^{k}}\right)-\left(-\frac{1}{e^{1}}\right) = -\left(\lim_{k\to\infty}\frac{1}{e^{k}}\right)+\frac{1}{e^{1}}$$

As k grows to infinity, $\frac{1}{e^{k}}$ approaches zero and $\lim_{k\to\infty}\frac{1}{e^{k}} = 0$. Therefore,

$$\int_{x=1}^{\infty} e^{-x}\,dx = 0 + \frac{1}{e^{1}} = \frac{1}{e} \approx 0.368$$

Explanation: We have used calculator to estimate the numerical value.

2

$$\int_{x=-1}^{1} \cosh x\,dx = [\sinh x]_{x=-1}^{1} = \sinh(1) - \sinh(-1)$$

$$= \frac{e^{1}-e^{-1}}{2} - \frac{e^{-1}-e^{-(-1)}}{2}\frac{e^{1}-e^{-1}}{2} - \frac{e^{-1}-e^{1}}{2} = \frac{e^{1}-e^{-1}}{2} - \left[-\left(\frac{e^{1}-e^{-1}}{2}\right)\right]$$

$$= \frac{e^{1}-e^{-1}}{2} + \frac{e^{1}-e^{-1}}{2} = 2\left(\frac{e^{1}-e^{-1}}{2}\right) = 2\sinh(1) \approx 2.350$$

Explanation:

- Recall that $\sinh x = \frac{e^{x}-e^{-x}}{2}$.
- Both $\sinh(1) - \sinh(-1)$ and $2\sinh(1)$ are correct answer. We have used the calculator to get the numerical value of the answer.

Set B

3

$$\int_{x=1}^{e} (\ln x + x)\,dx = \int_{x=1}^{e} \ln x\,dx + \int_{x=1}^{e} x\,dx = [x\ln x - x]_{x=1}^{e} + \left[\frac{x^2}{2}\right]_{x=1}^{e}$$

$$= (e\ln e - e) - (1\ln 1 - 1) + \left[\frac{e^2}{2} - \frac{1^2}{2}\right] = e(1) - e - [1(0) - 1] + \left[\frac{e^2}{2} - \frac{1}{2}\right]$$

$$= (e - e) - (0 - 1) + \left(\frac{e^2 - 1}{2}\right) = (0) - (-1) + \left(\frac{e^2 - 1}{2}\right) = 1 + \frac{e^2 - 1}{2}$$

$$= \frac{2}{2} + \frac{e^2 - 1}{2} = \frac{2 + e^2 - 1}{2} = \frac{1 + e^2}{2} \approx 4.195$$

Explanation:

- Recall that when the exponent n of x is $n = -1$, we have $\int x^{-1}\,dx = \ln x + c$. We drop the arbitrary constant c because it gets cancelled when we subtract after evaluating the antiderivative at upper and lower limits.
- $\ln e = 1$ and $\ln 1 = 0$. We used the calculator to get the numerical value.

4

$$\int_{x=1}^{2} 4^x\,dx = \left[\frac{4^x}{\ln 4}\right]_{x=1}^{2} = \frac{4^2}{\ln 4} - \frac{4^1}{\ln 4} = \frac{16 - 4}{\ln 4} = \frac{12}{\ln 4} \approx 8.656$$

Check: We will check that $\left(\frac{4^x}{\ln 4}\right)$ is the anti-derivative of 4^x.

$$\frac{d}{dx}\frac{4^x}{\ln 4} = \frac{1}{\ln 4}\frac{d}{dx}4^x = \frac{1}{\ln 4}(4^x \ln 4) = 4^x.$$

Explanation: We used the calculator to get the numerical value of the answer.

$$\int_{x=0}^{1} (x + \sinh x)\,dx = \int_{x=0}^{1} x\,dx + \int_{x=0}^{1} \sinh x\,dx = \left[\frac{x^2}{2}\right]_{x=0}^{1} + [\cosh x]_{x=0}^{1}$$

$$= \left[\frac{1^2}{2} - \frac{0^2}{2}\right] + (\cosh(1) - \cosh 0) = \frac{1}{2} + \cosh(1) - 1 = \frac{1}{2} - 1 + \cosh(1)$$

$$= -\frac{1}{2} + \cosh(1) = -\frac{1}{2} + \frac{2\cosh(1)}{2} = \frac{-1 + 2\cosh(1)}{2} \approx 1.043$$

Explanation:

- Note that $\cosh 0 = 1$. Recall that the hyperbolic cosine function is defined as $\cosh x = \frac{e^x + e^{-x}}{2}$.
- We have used the calculator to get the numerical value of the answer.

18 INTEGRATION BY POLYNOMIAL SUBSTITUTION

In this chapter, we will learn how to evaluate integrals by changing the variable of integration from x to another variable u. Such change of variable method, which also goes by the name of integration by substitution, is one of the most important techniques available to us for performing the integrals. For using this method, we follow the following steps.

1. First step is to intelligently guess a useful substitution which defines the new variable u as a function of x. In the present chapter, we will be choosing u as a polynomial function of x. In most cases, we choose a function whose derivative also appears in the integrand. We then have an equation that has variable u on one side and variable x on the other side.
2. Take the derivative of u with respect to x. From this, we obtain an equation, which expresses du in terms of dx.
3. From step 2, we can solve for dx in terms of du.
4. Finally substitute the values obtained in steps 1 to 3 to rewrite $\int f(x)\ dx$ in the form $\int g(u)\ du$.
5. For evaluating a definite integral $\int_a^b f(x)\ dx$, follow the steps 1 to step 4. In addition, we need to find the new limits for the variable u. For this, make use of the relation between u and x from step 1. Substitute the lower limit $x = a$ in this relation to get the lower limit u_1. Then substitute the upper limit $x = b$ in the relation to get the upper limit u_2. We will thus get the new integral of the form shown below that can be evaluated.

$$\int_{x=a}^{b} f(x)\ dx = \int_{u=u_1}^{u_2} g(u)\ du$$

Integration by polynomial substitution discussed above works only if after following steps 1 to step 4, we are able to replace all the terms and the new integral is completely expressible in terms of new variable u and there is no further dependence on x. In practice, experience gained by solving a number of problems involving use of this method helps us in choosing the right substitution.

Example 1: Evaluate the following integral

$$\int (5x + 4)^4\ dx$$

Solution: Let us make the following substitution $u = 5x + 4$. Then,

$\frac{du}{dx} = 5 + 0 = 5$, which gives $du = 5dx$ or $dx = du/5$

$$\int (5x + 4)^4\, dx = \int u^4 \left(\frac{du}{5}\right) = \int \frac{u^4}{5}\, du = \frac{u^{4+1}}{5(4+1)} + c = \frac{u^5}{5(5)} + c$$

$$= \frac{(5x + 4)^5}{25} + c$$

Example 2: Evaluate the following integral

$$\int_{x=-1}^{1} x^4 \sqrt{x^5 + 1}\, dx$$

Solution: We make the substitution $u = x^5 + 1$. Then,

$\frac{du}{dx} = 5x^4 + 0 = 5x^4$, which gives $du = 5x^4 dx$ or $dx = \frac{du}{5x^4}$

Note that, when $x = -1,\ u = (-1)^5 + 1 = -1 + 1 = 0.$

When $x = 1,\ u = (1)^5 + 1 = 1 + 1 = 2.$

Thus, when x varies from -1 to 1, u varies from 0 to 2 and

$$\int_{x=-1}^{1} x^4 \sqrt{x^5 + 1}\, dx = \int_{u=0}^{2} x^4 \sqrt{u} \left(\frac{du}{5x^4}\right) = \frac{1}{5} \int_{u=0}^{2} \sqrt{u}\, du = \int_{u=0}^{2} \frac{u^{1/2}}{5}\, du$$

$$= \left[\frac{u^{1/2+1}}{5(1/2 + 1)}\right]_{u=0}^{2} = \left[\frac{u^{3/2}}{5\left(\frac{3}{2}\right)}\right]_{u=0}^{2} = \left[\frac{2u^{3/2}}{15}\right]_{u=0}^{2} = \frac{2(2)^{3/2}}{15} - \frac{2(0)^{3/2}}{3}$$

$$= \frac{2(2)^{3/2}}{15} - 0 = \frac{2\left[2(2)^{1/2}\right]}{15} = \frac{4\sqrt{2}}{15} \approx 0.377$$

Note that we have used the following results:

- $\sqrt{u} = u^{\frac{1}{2}},\ \frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2},\ 5 \times \frac{3}{2} = \frac{15}{2}$, and $\frac{1}{\frac{15}{2}} = 1 \times \frac{2}{15} = \frac{2}{15}$.
- Since $x^{m+n} = x^m x^n$, we have $2^{3/2} = 2^{1+\frac{1}{2}} = 2^1 2^{1/2} = 2\sqrt{2}$.

Example 3: Evaluate the following integral

$$\int_{x=0}^{\pi/12} 8\tan 3x \, dx$$

Solution: We make the following substitution

$$u = 3x$$

Then, $\frac{du}{dx} = 3,$ which gives $du = 3dx$ or $dx = \frac{du}{3}$

Note that, when $x = 0,\ u = 3(0) = 0.$

When $x = \pi/12,\ u = 3\left(\frac{\pi}{12}\right) = \pi/4.$

Thus, when x varies from 0 to $\frac{\pi}{12}$, u varies from 0 to $\frac{\pi}{4}$ and

$$\int_{x=0}^{\pi/12} 8\tan 3x \, dx = 8\int_{x=0}^{\pi/12} \tan 3x \, dx = 8\int_{u=0}^{\pi/4} \tan u \left(\frac{du}{3}\right) = \frac{8}{3}\int_{u=0}^{\pi/4} \tan u du$$

$$= \frac{8}{3}[\ln|\sec u|]_{u=0}^{\pi/4} = \frac{8}{3}\left[\ln\left|\sec\frac{\pi}{4}\right| - \ln|\sec 0|\right]$$

$$= \frac{8}{3}[\ln\sqrt{2} - \ln 1] = \frac{8}{3}[\ln\sqrt{2} - 0] = \frac{8}{3}(\ln\sqrt{2})$$

$$= \frac{8}{3}(\ln 2^{1/2}) = \frac{8}{3}\left(\frac{1}{2}\ln 2\right) = \frac{4\ln 2}{3} \approx 0.924$$

Note that we have used the following results.

- $\frac{\pi}{4}$ radians $= \frac{180^0}{4} = 45^0, \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \ \sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2},$ and $\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1.$
- $\ln(1) = 0,\ \ \ln 2^{1/2} = \frac{1}{2}\ln 2,\ $ since $\ln x^n = n\ln x.$

Chapter 18 Exercises

Set A

Evaluate the following integrals.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\int \frac{1}{(x - 4)^3}\, dx =$$

[redacted]

$$\int \frac{1}{\sqrt{3x + 4}}\, dx =$$

[redacted]

$$\int_{x = 0}^{\pi/2} 4\cos 2x\, dx =$$

Set B

4

$$\int \sqrt{3x + 2}\, dx =$$

5 [redacted]

$$\int \frac{3x^3}{(x^4 + 5)}\,dx =$$

Set C

6

$$\int 8x^4 \sqrt{x^5 + 3}\,dx =$$

7

$$\int \frac{4x^2}{(2x^3 + 7)^2}\,dx$$

Set D

8

$$\int \sqrt{4 + \sqrt{x}}\,dx =$$

$$\int_{x=0}^{1} xe^{x^2}\,dx =$$

Set E

10

$$\int_{x=0}^{1} \frac{1}{2 + 3x}\,dx =$$

11

$$\int_{x=0}^{\sqrt{\pi/2}} x\sin x^2\,dx =$$

Chapter 18 Solutions

Set A

1

$$\int \frac{1}{(x-4)^3}\,dx = ?$$

Let $u = x - 4$. Then $\frac{du}{dx} = 1$ or which gives $du = dx$ or $dx = du$

$$\int \frac{1}{(x-4)^3}\,dx = \int \frac{1}{u^3}(du) = \int u^{-3}\,du$$

$$= \left(\frac{u^{-3+1}}{-3+1}\right) + c = \left(\frac{u^{-2}}{-2}\right) + c$$

$$= -\frac{1}{2u^2} + c = -\frac{1}{2(x-4)^2} + c$$

2

$$\int \frac{1}{\sqrt{3x+4}}\,dx = ?$$

Let $u = 3x + 4$. Then $\frac{du}{dx} = 3$, which gives $du = 3dx$ or $dx = \frac{du}{3}$

$$\int \frac{1}{\sqrt{3x+4}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{du}{3}\right) = \int \frac{1}{u^{1/2}}\left(\frac{du}{3}\right) = \frac{1}{3}\int u^{-1/2}\,du = \left(\frac{1}{3}\right)\left(\frac{u^{-1/2+1}}{-\frac{1}{2}+1}\right) + c$$

$$= \left(\frac{1}{3}\right)\left(\frac{u^{1/2}}{1/2}\right) + c = \left(\frac{2}{3}\right)(u^{1/2}) + c = \left(\frac{2}{3}\right)(3x+4)^{1/2} + c = \frac{2\sqrt{3x+4}}{3} + c$$

Explanation:

- $\sqrt{u} = u^{1/2}$, $\frac{1}{u^{1/2}} = u^{-1/2}$.
- $-\frac{1}{2} + 1 = -\frac{1}{2} + \frac{2}{2} = \frac{-1+2}{2} = \frac{1}{2}$,
- $\frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$.

3

$$\int_{x=0}^{\pi/4} 4\cos 2x \, dx = ?$$

Let $u = 2x$. Then $\frac{du}{dx} = 2$, which gives $du = 2dx$ or $dx = \frac{du}{2}$

Note that, when $x = 0$, $u = 2(0) = 0$ and when $x = \frac{\pi}{4}$, $u = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$.

Thus, when x varies from 0 to $\pi/4$, u varies from 0 to $\pi/2$ and

$$\int_{x=0}^{\pi/4} 4\cos 2x \, dx = \int_{u=0}^{\pi/2} 4\cos u \left(\frac{du}{2}\right) = \int_{u=0}^{\pi/2} 2\cos u du$$

$$= 2[\sin u]_{u=0}^{\pi/2} = 2\left(\sin\frac{\pi}{2} - \sin 0\right) = 2(1 - 0) = 2(1) = 2$$

Explanation:

- $\frac{\pi}{2}$ radians $= \frac{180^0}{2} = 90^0$, $\sin\frac{\pi}{2} = 1$ and $\sin 0 = 0$.

Set B

4

$$\int \sqrt{3x + 2} \, dx = ?$$

Let $u = 3x + 2$. Then $\frac{du}{dx} = 3$, which gives $du = 3dx$ or $dx = \frac{du}{3}$

$$\int \sqrt{3x + 2} \, dx = \int \sqrt{u} \left(\frac{du}{3}\right) = \int u^{1/2} \frac{du}{3} = \frac{1}{3}\int u^{1/2} \, du = \frac{1}{3}\left(\frac{u^{1/2 + 1}}{\frac{1}{2} + 1}\right) + c$$

$$= \frac{1}{3}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + c = \frac{2}{9}\left(u^{3/2}\right) + c = \frac{2}{9}(3x + 2)^{3/2} + c = \frac{2(3x + 2)^{3/2}}{9} + c$$

Explanation: $\sqrt{u} = u^{1/2}$, $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1 + 2}{2} = \frac{3}{2}$.

- $\frac{1}{3} \div \frac{3}{2} = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$.

$$\int \frac{3x^3}{(x^4+5)}\,dx = ?$$

Let $u = x^4 + 5$. Then $\frac{du}{dx} = 4x^3$, which gives $du = 4x^3dx$ or $dx = \frac{du}{4x^3}$

$$\int \frac{3x^3}{(x^4+5)}\,dx = \int \frac{3x^3}{(u)}\left(\frac{du}{4x^3}\right) = \frac{3}{4}\int \frac{x^3}{u}\frac{du}{x^3} = \frac{3}{4}\int u^{-1}\,du = \frac{3}{4}\ln u + c$$

$$= \frac{3}{4}\ln\left(x^4+5\right) + c$$

Explanation:

- Recall that when the exponent n of x is n =- 1, we have $\int u^{-1}\,dx = \ln u + c$.

Set C

6

$$\int 8x^4\sqrt{x^5+3}\,dx = ?$$

Let $u = x^5 + 3$. Then $\frac{du}{dx} = 5x^4$, which gives $du = 5x^4dx$ or $dx = \frac{du}{5x^4}$

$$\int 8x^4\sqrt{x^5+3}\,dx = \int 8x^4\sqrt{u}\left(\frac{du}{5x^4}\right) = \frac{8}{5}\int\sqrt{u}\,du = \frac{8}{5}\int u^{1/2}\,du$$

$$= \frac{8}{5}\left(\frac{u^{1/2+1}}{\frac{1}{2}+1}\right) + c = \frac{8}{5}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + c = \left(\frac{8}{5}\right)\left(\frac{2}{3}\right)u^{3/2} + c = \frac{16\left(x^5+3\right)^{3/2}}{15} + c$$

Explanation:

- $\sqrt{u} = u^{1/2}$.
- $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$, $\frac{8}{5} \div \frac{3}{2} = \frac{8}{5} \times \frac{2}{3} = \frac{16}{15}$.

7

Let $u = 2x^3 + 7$. Then $\frac{du}{dx} = 6x^2$, $du = 6x^2dx$ or $dx = \frac{du}{6x^2}$

$$\int \frac{4x^2}{(2x^3+7)^2}\,dx = \int \frac{4x^2}{u^2}\left(\frac{du}{6x^2}\right) = \frac{4}{6}\int \frac{du}{u^2} = \frac{4}{6}\int u^{-2}\,du = \frac{2}{3}\left(\frac{u^{-2+1}}{-2+1}\right) + c$$

$$= \frac{2}{3}\left(\frac{u^{-1}}{-1}\right) + c = -\frac{2}{3u} + c = -\frac{2}{3(2x^3 + 7)} + c$$

Explanation:

- $-2 + 1 = -1, \quad u^{-1} = \frac{1}{u}.$

8

$$\int \sqrt{4 + \sqrt{x}}\ dx = ?$$

Let $u = 4 + \sqrt{x}$. Then, $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, or $du = \frac{dx}{2\sqrt{x}}$ or $dx = 2\sqrt{x}\ du.$

Note that $u = 4 + \sqrt{x}$ gives $\sqrt{x} = u - 4.$

$$\int \sqrt{4 + \sqrt{x}}\ dx = \int \sqrt{u}\ (2\sqrt{x}\ du) = 2\int \sqrt{u}\ \sqrt{x}\ du = 2\int \sqrt{u}\ (u - 4)\ du$$

$$= 2\int (\sqrt{u}u - 4\sqrt{u})\ du = 2\int u^{1/2}u\ du - 2\int 4\sqrt{u}\ du$$

$$= 2\int u^{3/2}\ du - 8\int u^{1/2}\ du = 2\left(\frac{u^{3/2 + 1}}{\frac{3}{2} + 1}\right) - 8\left(\frac{u^{1/2 + 1}}{1/2 + 1}\right) -+ c$$

$$= 2\left(\frac{u^{5/2}}{\frac{5}{2}}\right) - 8\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + c = (2)\left(\frac{2}{5}\right)u^{5/2} - (8)\left(\frac{2}{3}\right)u^{3/2} + c$$

$$= \frac{4u^{5/2}}{5} - \frac{16u^{3/2}}{3} + c = \frac{4(4 + \sqrt{x})^{5/2}}{5} - \frac{16(4 + \sqrt{x})^{3/2}}{3} + c$$

Explanation:

- $\sqrt{u} = u^{1/2}, \ u^{1/2 + 1}u^1 = u^{1/2 + 1} = u^{3/2}.$
- $\frac{3}{2} + 1 = \frac{3}{2} + \frac{2}{2} = \frac{3 + 2}{2} = \frac{5}{2}, \ \frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1 + 2}{2} = \frac{3}{2}, \ 2 \div \frac{5}{2} = 2 \times \frac{2}{5}, \ 8 \div \frac{3}{2} = 8 \times \frac{2}{3}.$

9

$\int_{x=0}^{1} xe^{x^2}\ dx = ?$ Let $u = x^2$. Then $\frac{du}{dx} = 2x, \ du = 2xdx$ or $dx = \frac{du}{2x}$

Note that, when $x = 0, \ u = (0)^2 = 0$ and when $x = 1, \ u = (1)^2 = 1.$

Thus, when x varies from 0 to 1, u varies from 0 to 1 and

$$\int_{x=0}^{1} xe^{x^2}\, dx$$

$$= \int_{u=0}^{1} xe^{u}\left(\frac{du}{2x}\right) = \int_{u=0}^{1} \frac{xe^{u}du}{2x} = \frac{1}{2}\int_{u=0}^{1} e^{u}\, du = \left[\frac{1}{2}(e^{u})\right]_{u=0}^{1}$$

$$= \left[\frac{e^{u}}{2}\right]_{u=0}^{1} = \frac{e^{1}}{2} - \frac{e^{0}}{2} = \frac{e}{2} - \frac{1}{2} = \frac{e-1}{2} \approx 0.859$$

Explanation: $e^{0} = 1$. We have used calculator to estimate that $\frac{e-1}{2} \approx 0.8591$.

Set E

10

$$\int_{x=0}^{1} \frac{1}{2+3x}\, dx = ?$$

Let $u = 2 + 3x$. Then, $\frac{du}{dx} = 0 + 3 = 3$, which gives $du = 3dx$ or $dx = \frac{du}{3}$

Note that, when $x = 0,\ u = 2 + 3(0) = 2$ and when $x = 1,\ u = 2 + 3(1) = 5$.

Thus, when x varies from 0 to 1, u varies from 2 to 5 and

$$\int_{x=0}^{1} \frac{1}{2+3x}\, dx = \int_{u=2}^{5} \frac{1}{u}\left(\frac{du}{3}\right) = \frac{1}{3}\int_{u=2}^{5} \frac{1}{u}\, du = \frac{1}{3}\int_{u=2}^{5} u^{-1}\, du = \frac{1}{3}[\ln u]_{u=2}^{5}$$

$$= \frac{1}{3}[\ln 5 - \ln 2] = \frac{1}{3}\left[\ln\left(\frac{5}{2}\right)\right] = \frac{1}{3}[\ln(2.5)] \approx 0.305$$

Explanation:

- Recall that $\int x^{-1}\, dx = \ln x + c$. We drop the arbitrary constant c because it gets cancelled when we subtract after evaluating the antiderivative at upper and lower limits.
- Since $\ln x - \ln y = \ln\frac{x}{y}$, $\ln 5 - \ln 2 = \ln\left(\frac{5}{2}\right)$.
- We have used calculator to estimate that $\frac{1}{3}[\ln(2.5)] \approx 0.305$.

$$\int_{x=0}^{\sqrt{\pi}/2} x\sin x^2 \, dx = ?$$

Let $u = x^2$. Then, $\frac{du}{dx} = 2x$, which gives $du = 2xdx$ or $dx = \frac{du}{2x}$

Note that, when $x = 0,\ u = (0)^2 = 0$ and when $x = \frac{\sqrt{\pi}}{2},\ u = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}$.

Thus, when x varies from 0 to $\frac{\sqrt{\pi}}{2}$, u varies from 0 to $\frac{\pi}{4}$ and

$$\int_{x=0}^{\sqrt{\pi}/2} x\sin x^2 \, dx = \int_{u=0}^{\pi/4} x\sin u \left(\frac{du}{2x}\right) = \frac{1}{2}\int_{u=0}^{\pi/4} \sin u du = \frac{1}{2}[-\cos u]_{u=0}^{\pi/4}$$

$$= \frac{1}{2}\left[-\cos\frac{\pi}{4} - (-\cos 0)\right] = \frac{1}{2}\left[-\left(\frac{\sqrt{2}}{2}\right) + 1\right] = -\frac{\sqrt{2}}{4} + \frac{1}{2} = -\frac{\sqrt{2}}{4} + \frac{2}{4} = \frac{-\sqrt{2}+2}{4} \approx 0.146$$

Explanation:

- $\frac{\pi}{4}$ radians $= \frac{180^0}{4} = 45^0$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $\cos 0 = 1$

19 INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

In the present chapter, we continue with the substitution method for evaluation of integrals. The method of integration by trigonometric substitution discussed in the present chapter is particularly useful in the following cases.

1. Integrals involving trigonometric functions like the following:

   $$\int \frac{\sin\theta}{\cos^4\theta}\, d\theta\,,\quad \int \sin^3\theta\, d\theta\,,\quad \int \sin\theta\cos\theta\, d\theta\,,\quad \int \tan^3\theta\, d\theta\,,\quad \int \cos^4 x\, dx$$

   For evaluating such integrals, usually the following substitutions work:

   - If the power of sine function is odd and positive, substitute $u = \cos\theta$ and use the result $\sin^2\theta = 1 - \cos^2\theta$ (which follows from the identity $\sin^2\theta + \cos^2\theta = 1)$ to convert the remaining sines to cosines.
   - If the power of cosine function is odd and positive, substitute $u = \sin\theta$ and use the result $\cos^2\theta = 1 - \sin^2\theta$ (which follows from the same identity $\sin^2\theta + \cos^2\theta = 1)$ to convert the remaining cosines to sines.
   - If the power of both sine and cosine functions is odd, we can make either of the above substitutions, depending on which of them is simplifies the integral.

     Having made the substitution, find the corresponding derivative $\frac{du}{d\theta}$ and rearrange the resulting equation to express $d\theta$ in terms of du.
   - If the power of both sine and cosine functions is even, we first use the relations $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ or $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ (both of which follow from the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$) to first convert the integrand to odd powers of cosines of 2θ. We then make the appropriate substitutions and proceed as above.
   - Finally, for functions involving $\tan\theta$ and $\sec\theta$, we make the substitutions $u = \tan\theta$ and $u = \sec\theta$ with the corresponding derivatives $\frac{d}{d\theta}\tan\theta = \sec^2\theta$ and $\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$ and use the identity $1 + \tan^2\theta = \sec^2\theta$. Then general considerations similar to those involving $\sin\theta$ and $\cos\theta$ work very well.

2. Integrals involving expressions of the type $(a^2 - x^2)^n$, where n is any real number.

   In this case, we will make the substitution $x = a\sin\theta$ so that $\theta = \sin^{-1} x/a$. Then

$\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta d\theta$. We also have $a^2 - x^2 = a^2 - a^2\sin^2\theta$ $= a^2(1 - \sin^2\theta) = a^2\cos^2\theta$, because $\sin^2\theta + \cos^2\theta = 1$.

As an example, consider $\int \frac{1}{\sqrt{25 - x^2}} dx = \int \frac{1}{\sqrt{5^2 - x^2}} dx = \int (5^2 - x^2)^{-1/2} dx.$

Here, the integrand contains the expression $(5^2 - x^2)^{-1/2}$. So, $n = -1/2$, $a = 5$.

3. Integrals involving expressions of the type $(a^2 + x^2)^n$, where n is any real number.

   In this case, we will make the substitution $x = a\tan\theta$ so that $\theta = \tan^{-1} x/a$. Then $\frac{dx}{d\theta} = a\sec^2\theta$ and $dx = a\sec^2\theta d\theta$. We also have $a^2 + x^2 = a^2 + a^2\tan^2\theta$ $= a^2(1 + \tan^2\theta) = a^2\sec^2\theta$, because $1 + \tan^2\theta = \sec^2\theta$.

   As an example, consider $\int \frac{1}{9 + x^2} dx = \int \frac{1}{3^2 + x^2} dx = \int (3^2 + x^2)^{-1} dx.$

   Here, the integrand contains the expression $(3^2 + x^2)^{-1}$. So, $n = -1$, $a = 3$.

4. Integrals involving expressions of the type $(x^2 - a^2)^n$, where n is any real number.

   In this case, we will make the substitution $x = a\sec\theta$ so that $\theta = \sec^{-1} x/a$. Then $\frac{dx}{d\theta} = a\sec\theta\tan\theta$ and $dx = a\sec\theta\tan\theta d\theta$. We also have $x^2 - a^2 = a\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$, because $1 + \tan^2\theta = \sec^2\theta$.

   As an example, consider $\int \sqrt{x^2 - 16}\, dx = \int \sqrt{x^2 - 4^2}\, dx = \int (x^2 - 4^2)^{1/2} dx.$

   Here, the integrand contains the expression $(x^2 - 4^2)^{1/2}$. So, $n = \frac{1}{2}$, $a = 4$.

Let us recall the following important relations and identities, which we will be making use of in this chapter:

$$\sin\theta = \frac{1}{\csc\theta} \text{ or } \csc\theta = \frac{1}{\sin\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \text{ or } \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Example 1: Evaluate the following integral

$$\int_{\theta=0}^{\pi/2} 4\sin^5\theta\cos\theta d\theta$$

Solution: We make the substitution $u = \sin\theta$

$$\text{Then, } \frac{du}{d\theta} = \cos\theta, \text{ which gives } du = \cos\theta d\theta \text{ or } d\theta = \frac{du}{\cos\theta}$$

Note that, when $\theta = 0, u = \sin 0 = 0.$ When $\theta = \frac{\pi}{2}, u = \sin\frac{\pi}{2} = 1.$

Thus, when θ varies from 0 to $\pi/2$, u varies from 0 to 1 and

$$\int_{\theta=0}^{\pi/2} 4\sin^5\theta\cos\theta dx = \int_{u=0}^{1} 4u^5\cos\theta\left(\frac{du}{\cos\theta}\right) = \int_{u=0}^{1} 4u^5 du = 4\int_{u=0}^{1} u^5 du$$

$$= 4\left[\frac{u^{5+1}}{5+1}\right]_{u=0}^{1} = 4\left[\frac{u^6}{6}\right]_{u=0}^{1} = \frac{2}{3}[u^6]_{u=0}^{1} = \frac{2}{3}(1^6 - 0^6) = \frac{2}{3}(1-0) = \frac{2}{3} \approx 0.667$$

Example 2: Evaluate the following integral

$$\int\frac{1}{\sqrt{25 - x^2}}\, dx$$

Solution: We have

$$\int\frac{1}{\sqrt{25 - x^2}}\, dx = \int\frac{1}{\sqrt{5^2 - x^2}}\, dx = \int(5^2 - x^2)^{-1/2}\, dx$$

Comparing with the form $(a^2 - x^2)^n,\ n = -1/2, a = 5.$ Put $x = 5\sin\theta$ so that $\theta = \sin^{-1}\frac{x}{5}$

Then, $\frac{dx}{d\theta} = 5\cos\theta,$ which gives $dx = 5\cos\theta d\theta$

$$\int\frac{1}{\sqrt{5^2 - x^2}}\, dx = \int\frac{1}{\sqrt{5^2 - (5\sin\theta)^2}}(5\cos\theta d\theta) = \int\frac{5\cos\theta d\theta}{\sqrt{5^2 - 5^2\sin^2\theta}}$$

$$= \int\frac{5\cos\theta d\theta}{\sqrt{5^2(1 - \sin^2\theta)}} = \int\frac{5\cos\theta d\theta}{\sqrt{5^2\cos^2\theta}} = \int\frac{5\cos\theta d\theta}{5\cos\theta} = \int d\theta = \int\theta^0\, d\theta$$

$$= \left(\frac{\theta^{0+1}}{0+1}\right) + c = \left(\frac{\theta}{1}\right) + c = \theta + c = \sin^{-1}\frac{x}{5} + c$$

Note that we have written $1 - \sin^2\theta = \cos^2\theta$ because $\sin^2\theta + \cos^2\theta = 1$.

Example 3: Evaluate the following integral

$$\int_{\theta=0}^{\pi/4} \tan^4\theta \, d\theta$$

Solution: $\displaystyle\int_{\theta=0}^{\pi/4} \tan^4\theta \, d\theta = \int_{\theta=0}^{\pi/4} \tan^2\theta \tan^2\theta d\theta = \int_{\theta=0}^{\pi/4} \tan^2\theta \left(\sec^2\theta - 1\right) d\theta$

$$= \int_{\theta=0}^{\pi/4} \left(\tan^2\theta\sec^2\theta - \tan^2\theta\right) d\theta$$

$$= \int_{\theta=0}^{\pi/4} \tan^2\theta\sec^2\theta \, d\theta - \int_{\theta=0}^{\pi/4} \tan^2\theta \, d\theta = \int_{\theta=0}^{\pi/4} \tan^2\theta\sec^2\theta \, d\theta - \int_{\theta=0}^{\pi/4} \tan^2\theta \, d\theta$$

Now, in order to evaluate $\int_0^{\pi/4} \tan^2\theta\sec^2\theta \, d\theta,$ we make the following substitution

$u = \tan\theta$ Then, $\frac{du}{d\theta} = \sec^2\theta,$ which gives $du = \sec^2\theta d\theta$ or $d\theta = \frac{du}{\sec^2\theta}$

Note that, when $\theta = 0, \ u = \tan 0 = 0.$ When $\theta = \pi/4, \ u = \tan\frac{\pi}{4} = 1.$

Thus, when θ varies from 0 to $\pi/4$, u varies from 0 to 1 and

$$\int_{\theta=0}^{\pi/4} \tan^4\theta \, d\theta = \int_{u=0}^{1} u^2\sec^2\theta\left(\frac{du}{\sec^2\theta}\right) - \int_{\theta=0}^{\pi/4} \left(\sec^2\theta - 1\right) d\theta$$

$$= \int_{u=0}^{1} u^2 du - \int_{\theta=0}^{\pi/4} \left(\sec^2\theta - 1\right) d\theta = \int_{u=0}^{1} u^2 du - \int_{\theta=0}^{\pi/4} \left(\sec^2\theta - 1\right) d\theta$$

$$= \left[\frac{u^3}{3}\right]_{u=0}^{1} - \int_0^{\frac{\pi}{4}} \sec^2\theta \, d\theta + \int_0^{\frac{\pi}{4}} \theta^0 \, d\theta = \left(\frac{1^3}{3} - \frac{0^3}{3}\right) - \left[\tan\theta\right]_{\theta=0}^{\pi/4} + \left[\frac{\theta^{0+1}}{0+1}\right]_{\theta=0}^{\pi/4}$$

$$= \left(\frac{1}{3} - 0\right) - \left(\tan\frac{\pi}{4} - \tan 0\right) + \left[\frac{\theta^1}{1}\right]_{\theta=0}^{\pi/4} = \left(\frac{1}{3}\right) - (1 - 0) + \left[\theta\right]_{\theta=0}^{\pi/4}$$

$$= \frac{1}{3} - 1 + \left(\frac{\pi}{4} - 0\right) = \frac{1}{3} - \frac{1}{1} + \frac{\pi}{4} = \frac{4 - 12 + 3\pi}{12} = \frac{-8 + 3\pi}{12} = \frac{3\pi - 8}{12} \approx 0.119$$

Note that we have used the result $\tan^2\theta = \sec^2\theta - 1$, because $1 + \tan^2\theta = \sec^2\theta$. Also, $\frac{\pi}{4}$ radians $= \frac{180}{4} = 45^0$, $\tan\frac{\pi}{4} = 1$, $\tan 0 = 0$ and $1 =$ θ^0.

Example 4: Evaluate the following integral

$$\int_{x=0}^{1} \frac{dx}{(x^2 + 1)}$$

Solution: We have

$$\int_{x=0}^{1} \frac{dx}{(x^2 + 1)} = \int_{x=0}^{1} \frac{dx}{(1^2 + x^2)} = \int_{x=0}^{1} (1^2 + x^2)^{-1} dx$$

Comparing with the form $(a^2 + x^2)^n, \ n = -1, a = 1.$ Put $x = 1\tan\theta = \tan\theta$ so that

$\theta = \tan^{-1}(x).$ Then, $\frac{dx}{d\theta} = \sec^2\theta,$ which gives $dx = \sec^2\theta d\theta$

Note that, when $x = 0, \theta = \tan^{-1}(0) = \tan^{-1}(0) = 0.$

When $x = 1, \theta = \tan^{-1}(1) = \frac{\pi}{4}.$

Thus, when x varies from 0 to 1, θ varies from 0 to $\frac{\pi}{4}$.

$$\int_{x=0}^{1} \frac{dx}{(1 + x^2)} = \int_{\theta=0}^{\pi/4} \frac{\sec^2\theta d\theta}{(1 + \tan^2\theta)} = \int_{\theta=0}^{\pi/4} \frac{\sec^2\theta d\theta}{1 + \tan^2\theta} = \int_{\theta=0}^{\pi/4} \frac{\sec^2\theta d\theta}{\sec^2\theta}$$

$$= \int_{\theta=0}^{\pi/4} d\theta = \int_{\theta=0}^{\pi/4} 1 d\theta = \int_{\theta=0}^{\pi/4} \theta^0 \, d\theta = \left[\frac{\theta^{0+1}}{0+1}\right]_{\theta=0}^{\pi/4} = \left[\frac{\theta}{1}\right]_{\theta=0}^{\pi/4}$$

$$= [\theta]_{\theta=0}^{\pi/4} = \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4} \approx 0.785$$

Note that we have used the following results:

- $1 + \tan^2\theta = \sec^2\theta.$
- $\frac{\pi}{4}$ radians $= \frac{180^0}{4} = 45^0, \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2},$
- $\cos 0 = 1.$
- Recall that $1 = \theta^0$ and $\theta^1 = \theta.$

Example 5: Evaluate the following integral

$$\int \frac{1}{(16 - x^2)^{3/2}}\, dx$$

Solution: We have

$$\int \frac{1}{(16 - x^2)^{3/2}}\, dx = \int \frac{1}{(4^2 - x^2)^{3/2}}\, dx = \int (4^2 - x^2)^{-3/2}\, dx$$

Comparing with the form $(a^2 - x^2)^n$, we have $n = -3/2$, $a = 4$. Put $x = 4\sin\theta$ so that $\sin\theta = \frac{x}{4}$, Then, , $\frac{dx}{d\theta} = 4\cos\theta$, which gives $dx = 4\cos\theta d\theta$

$$\int \frac{1}{(16 - x^2)^{3/2}}\, dx = \int \frac{1}{[4^2 - (4\sin\theta)^2]^{3/2}} (4\cos\theta d\theta)$$

$$= \int \frac{4\cos\theta d\theta}{[4^2 - 4^2\sin^2\theta]^{3/2}} = \int \frac{4\cos\theta d\theta}{[4^2(1 - \sin^2\theta)]^{3/2}} = \int \frac{4\cos\theta d\theta}{[4^2\cos^2\theta]^{3/2}}$$

$$= \int \frac{4\cos\theta d\theta}{(4^2)^{3/2}[\cos^2\theta]^{3/2}} = \int \frac{4\cos\theta d\theta}{(4)^3\cos^3\theta} = \int \frac{d\theta}{(4)^2\cos^2\theta} = \left(\frac{1}{4}\right)^2 \int \frac{d\theta}{\cos^2\theta}$$

$$= \left(\frac{1}{16}\right)\int \sec^2\theta d\theta = \left(\frac{1}{16}\right)\tan\theta + c = \left(\frac{1}{16}\right)\frac{\sin\theta}{\cos\theta} + c = \left(\frac{1}{16}\right)\frac{\frac{x}{4}}{\sqrt{1 - \sin^2\theta}} + c$$

$$= \left(\frac{1}{16}\right)\frac{\frac{x}{4}}{\sqrt{1 - \frac{x^2}{16}}} + c = \left(\frac{1}{16}\right)\frac{\frac{x}{4}}{\sqrt{\frac{16 - x^2}{16}}} + c$$

$$= \left(\frac{1}{16}\right)\left(\frac{x}{4}\right)\left(\frac{\sqrt{16}}{1}\right)\frac{1}{\sqrt{16 - x^2}} + c = \left(\frac{1}{16}\right)\left(\frac{x}{4}\right)\left(\frac{4}{1}\right)\frac{1}{\sqrt{16 - x^2}} + c = \frac{x}{16\sqrt{16 - x^2}} + c$$

Note that we have used the following results:

- $1 - \sin^2\theta = \cos^2\theta$ because $\sin^2\theta + \cos^2\theta = 1$.
- Since $(x^m)^n = x^{mn}$, we have $(4^2)^{3/2} = (4)^{2(3/2)} = (4)^3$.
- $\frac{1}{\cos^2\theta} = \sec^2\theta$.
- $\cos\theta = \sqrt{\cos^2\theta} = \sqrt{1 - \sin^2\theta}$ because $\sin^2\theta + \cos^2\theta = 1$.

Example 6: Evaluate the following integral

$$\int \frac{1}{\sqrt{x^2 - 25}}\, dx$$

Solution: We have

$$\int \frac{1}{\sqrt{x^2 - 25}}\, dx = \int \frac{1}{\sqrt{x^2 - 5^2}}\, dx = \int \left(x^2 - 5^2\right)^{-1/2} dx$$

Comparing with the form $(x^2 - a^2)^n$, we have $n =- 1/2$, $a = 5$.

Put $x = 5\sec\theta$ so that $\sec\theta = \frac{x}{5}$ or $\theta = \sec^{-1}\frac{x}{5}$ Then, $\frac{dx}{d\theta} = 5\sec\theta\tan\theta,$ which gives $dx = 5\sec\theta\tan\theta d\theta$

$$\int \frac{1}{\sqrt{x^2 - 25}}\, dx = \int \frac{1}{\sqrt{5^2\sec^2\theta - 5^2}}\,(5\sec\theta\tan\theta d\theta)$$

$$= \int \frac{1}{\sqrt{5^2(\sec^2\theta - 1)}}\,(5\sec\theta\tan\theta d\theta) = \int \frac{5\sec\theta\tan\theta d\theta}{5\sqrt{(\sec^2\theta - 1)}} = \int \frac{\sec\theta\tan\theta d\theta}{\sqrt{\tan^2\theta}}$$

$$= \int \frac{\sec\theta\tan\theta d\theta}{\tan\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c = \ln\left|\sec\theta + \sqrt{\sec^2\theta - 1}\right| + c$$

$$= \ln\left|\frac{x}{5} + \sqrt{\left(\frac{x}{5}\right)^2 - 1}\right| + c = \ln\left|\frac{x}{5} + \sqrt{\frac{x^2}{25} - 1}\right| + c = \ln\left|\frac{x}{5} + \sqrt{\frac{x^2 - 25}{25}}\right| + c$$

$$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2 - 25}}{5}\right| + c = \ln\left|\frac{x + \sqrt{x^2 - 25}}{5}\right| + c = \ln\left|x + \sqrt{x^2 - 25}\right| - \ln 5 + c$$

$$= \ln\left|x + \sqrt{x^2 - 25}\right| + c_1$$

Note that we have used the following results:

- $\sec^2\theta - 1 = \tan^2\theta$ because $1 + \tan^2\theta = \sec^2\theta.$ Also, $\tan\theta = \sqrt{\tan^2\theta} = \sqrt{\sec^2\theta - 1}$
- Because $-\ln 5$ is a constant, $-\ln 5 + c$ is also a constant, which we have denoted as c_1.

Chapter 19 Exercises

Set A

Evaluate the following integrals.
You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\int \frac{1}{\sqrt{25 - x^2}}\, dx =$$

2

$$\int \frac{1}{9 + x^2}\, dx =$$

3

$$\int \frac{\sin\theta}{\cos^4\theta}\, d\theta =$$

Set B

4

$$\int \frac{1}{(4 - x^2)^{3/2}} dx =$$

5

$$\int_{\theta = \pi/3}^{\pi/2} \sin^3\theta\, d\theta =$$

6

$$\int \frac{x^3}{9 + x^8}\, dx$$

Set C

7

$$\int_{x=0}^{\pi/2} \frac{3\sin x}{1+\cos^2 x}\,dx =$$

8

$$\int_{\theta=0}^{\pi/2} \sqrt{\sin\theta}\cos^3\theta\,dx =$$

$$\int \cos^4\theta\,dx =$$

Chapter 19 Solutions

Set A

1

$$\int \frac{1}{\sqrt{25 - x^2}}\,dx = \int \frac{1}{\sqrt{5^2 - x^2}}\,dx$$

Put $x = 5\sin\theta$ so that $\theta = \sin^{-1}\frac{x}{5}$. Then, $\frac{dx}{d\theta} = 5\cos\theta,$ which gives $dx = 5\cos\theta d\theta$

$$\int \frac{1}{\sqrt{5^2 - x^2}}\,dx = \int \frac{1}{\sqrt{5^2 - (5\sin\theta)^2}}(5\cos\theta d\theta) = \int \frac{5\cos\theta d\theta}{\sqrt{5^2 - 5^2\sin^2\theta}}$$

$$= \int \frac{5\cos\theta d\theta}{\sqrt{5^2(1 - \sin^2\theta)}} = \int \frac{5\cos\theta d\theta}{\sqrt{5^2\cos^2\theta}} = \int \frac{5\cos\theta d\theta}{5\cos\theta} = \int d\theta = \int \theta^0\,d\theta$$

$$= \left(\frac{\theta^{0+1}}{0+1}\right) + c = \left(\frac{\theta}{1}\right) + c = \theta + c = \sin^{-1}\frac{x}{5} + c$$

Explanation:

- $1 - \sin^2\theta = \cos^2\theta$because $\sin^2\theta + \cos^2\theta = 1.$

2

$$\int \frac{1}{9 + x^2}\,dx = \int \frac{1}{3^2 + x^2}\,dx$$

Put $x = 3\tan\theta$ so that $\tan\theta = \frac{x}{3}$ or $\theta = \tan^{-1}\frac{x}{3}$

Then, $\frac{dx}{d\theta} = 3\sec^2\theta$ which gives $dx = 3\sec^2\theta d\theta$

$$\int \frac{1}{3^2 + x^2}\,dx = \int \frac{1}{3^2 + 3^2\tan^2\theta}(3\sec^2\theta d\theta) = \int \frac{3\sec^2\theta d\theta}{3^2(1 + \tan^2\theta)}$$

$$= \int \frac{3\sec^2\theta d\theta}{3^2\sec^2\theta} = \int \frac{d\theta}{3} = \frac{1}{3}\int d\theta = \frac{1}{3}\int \theta^0\,d\theta = \frac{1}{3}\left(\frac{\theta^{0+1}}{0+1}\right) + c$$

$$= \frac{1}{3}\left(\frac{\theta^1}{1}\right) + c = \frac{1}{3}\theta + c = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$$

Explanation:

- Recall that we have the identity $1 + \tan^2\theta = \sec^2\theta.$

- Note that $1 = \theta^0$ and $\theta^1 = \theta$.

3 Let $u = \cos\theta$. Then, $\frac{du}{d\theta} = -\sin\theta$, which gives $du = -\sin\theta d\theta$ or $d\theta = \frac{du}{(-\sin\theta)}$

$$\int \frac{\sin\theta}{\cos^4\theta}d\theta = \int \frac{\sin\theta}{u^4}\frac{du}{(-\sin\theta)} = -\int \frac{du}{u^4} = -\int u^{-4}du = \left(\frac{u^{-4+1}}{-4+1}\right) + c$$

$$= -\left(\frac{u^{-3}}{-3}\right) + c = \frac{1}{3u^3} + c = \frac{1}{3\cos^3\theta} + c$$

Set B

4 We have

$$\int \frac{1}{(4 - x^2)^{3/2}}dx = \int \frac{1}{(2^2 - x^2)^{3/2}}dx$$

Put $x = 2\sin\theta$ so that $\sin\theta = \frac{x}{2}$. Then, $\frac{dx}{d\theta} = 2\cos\theta$, which gives $dx = 2\cos\theta d\theta$

$$\int \frac{1}{(2^2 - x^2)^{3/2}}dx = \int \frac{1}{(2^2 - (2\sin\theta)^2)^{3/2}}2\cos\theta d\theta = \int \frac{2\cos\theta}{(2^2 - 2^2\sin^2\theta)^{3/2}}d\theta$$

$$= \int \frac{2\cos\theta d\theta}{[2^2(1 - \sin^2\theta)]^{3/2}} = \int \frac{2\cos\theta d\theta}{[2^2\cos^2\theta]^{3/2}} = \int \frac{2\cos\theta d\theta}{2^3\cos^3\theta}$$

$$= \frac{2}{2^3}\int \frac{d\theta}{\cos^2\theta} = \frac{2}{8}\int \sec^2\theta d\theta = \frac{1}{4}\tan\theta + c = \frac{1\sin\theta}{4\cos\theta} + c$$

$$= \frac{\sin\theta}{4\sqrt{\cos^2\theta}} + c = \frac{\sin\theta}{4\sqrt{1 - \sin^2\theta}} + c = \frac{\frac{x}{2}}{4\sqrt{1 - \left(\frac{x}{2}\right)^2}} + c = \frac{x}{8\sqrt{1 - \frac{x^2}{4}}} + c = \frac{x}{4\sqrt{4 - x^2}} + c$$

Explanation:

- $2^2 - 2^2\sin^2\theta = 2^2(1 - \sin^2\theta) = 2^2\cos^2\theta$, since $\sin^2\theta + \cos^2\theta = 1$.
- Since $(x^m)^n = x^{mn}$, we have $[2^2\cos^2\theta]^{3/2} = (2^2)^{3/2}(\cos^2\theta)^{3/2} = 2^3\cos^3\theta$.

5

$$\int_{\theta=\pi/3}^{\pi/2} \sin^3\theta\, d\theta = \int_{\theta=\pi/3}^{\pi/2} \sin\theta \sin^2\theta\, d\theta = \int_{\theta=\pi/3}^{\pi/2} \sin\theta\left(1 - \cos^2\theta\right) d\theta$$

Now, let $u = \cos\theta$. Then, $\frac{du}{d\theta} = -\sin\theta$, $du = -\sin\theta d\theta$ or $d\theta = du/(-\sin\theta)$

Note that, when $\theta = \frac{\pi}{3}$, $u = \cos\frac{\pi}{3} = \frac{1}{2}$ and when $\theta = \frac{\pi}{2}$, $u = \cos\frac{\pi}{2} = 0.$

Thus, when θ varies from 0 to $\pi/2$, u varies from $\frac{1}{2}$ to 0. Therefore,

$$\int_{\theta=\pi/3}^{\pi/2} \sin^3\theta\, d\theta = \int_{\theta=\pi/3}^{\pi/2} \sin\theta\left(1 - \cos^2\theta\right) d\theta = \int_{u=1/2}^{0} \sin\theta\left(1 - u^2\right)\frac{du}{(-\sin\theta)}$$

$$= \int_{u=1/2}^{0} -\left(1 - u^2\right)du = \int_{u=1/2}^{0} \left(-1 + u^2\right)du = -\int_{u=1/2}^{0} u^0 du + \int_{u=1/2}^{0} u^2 du$$

$$= -\left[\frac{u^{0+1}}{0+1}\right]_{u=1/2}^{0} + \left[\frac{u^{2+1}}{2+1}\right]_{u=1/2}^{0} = -[u]_{u=1/2}^{0} + \frac{1}{3}\left[u^3\right]_{u=1/2}^{0}$$

$$= -\left(0 - \frac{1}{2}\right) + \frac{1}{3}\left[0^3 - \left(\frac{1}{2}\right)^3\right] = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{8}\right) = \frac{1}{2} - \frac{1}{24} = \frac{12}{24} - \frac{1}{24} = \frac{12-1}{24} = \frac{11}{24} \approx 0.458$$

Explanation: $\sin^2\theta = 1 - \cos^2\theta$, since $\sin^2\theta + \cos^2\theta = 1$.

- $\frac{\pi}{3}$ radians $= \frac{180^0}{3} = 60^0$, $\cos\frac{\pi}{3} = \frac{1}{2}$, $\frac{\pi}{2}$ radians $= \frac{180^0}{2} = 90^0$, $\cos\frac{\pi}{2} = 0.$

[redacted]Let $u = x^4$. Then, $\frac{du}{dx} = 4x^3$, which gives $du = 4x^3 dx$ or $dx = du/\left(4x^3\right)$

$$\int \frac{x^3}{9 + x^8}\, dx = \int \frac{x^3}{9 + \left(x^4\right)^2}\, dx = \int \frac{x^3}{9 + u^2} \frac{du}{\left(4x^3\right)} = \frac{1}{4}\int \frac{du}{9 + u^2} = \frac{1}{4}\int \frac{du}{3^2 + u^2}$$

Now, let $u = 3\tan\theta$ so that $\tan\theta = \frac{u}{3}$ or $\theta = \tan^{-1}\frac{u}{3} = \tan^{-1}\frac{x^4}{3}$

Then, $\frac{du}{d\theta} = 3\sec^2\theta$, which gives $du = 3\sec^2\theta d\theta$

$$\int \frac{x^3}{9 + x^8}\, dx = \frac{1}{4}\int \frac{du}{3^2 + u^2} = \frac{1}{4}\int \frac{3\sec^2\theta d\theta}{3^2 + 3^2\tan^2\theta} = \frac{1}{4}\int \frac{3\sec^2\theta d\theta}{3^2\left(1 + \tan^2\theta\right)}$$

$$= \frac{1}{4}\int \frac{3\sec^2\theta d\theta}{3^2\sec^2\theta} = \frac{1}{4}\int \frac{d\theta}{3} = \frac{1}{12}\int \theta^0\, d\theta = \frac{1}{12}\left(\frac{\theta^{0+1}}{0+1}\right) + c = \frac{\theta}{12} + c = \frac{1}{12}\tan^{-1}\frac{x^4}{3} + c$$

Explanation: Recall that we have the identity $1 + \tan^2\theta = \sec^2\theta$.

Set C

7

$$\int_{x=0}^{\pi/2} \frac{3\sin x}{1+\cos^2 x}\,dx = ?$$

Let $u = \cos x$. Then, $\frac{du}{dx} = -\sin x$, or $du = -\sin x dx$ or $dx = du/(-\sin x)$

Note that, when $x = 0,\ u = \cos 0 = 1$ and when $x = \frac{\pi}{2},\ u = \cos\frac{\pi}{2} = 0.$

Thus, when x varies from 0 to $\pi/2$, u varies from 1 to 0.

$$\int_{x=0}^{\pi/2} \frac{3\sin x}{1+\cos^2 x}\,dx = \int_{u=1}^{0} \frac{3\sin x}{1+u^2}\frac{du}{(-\sin x)} = -3\int_{u=1}^{0}\frac{du}{1+u^2}$$

Now, put $u = \tan\theta$ or $\theta = \tan^{-1} u$. Then, $\frac{du}{d\theta} = \sec^2\theta$, which gives $du = \sec^2\theta d\theta$

Note that, when $u = 1,\ \theta = \tan^{-1}(1) = \pi/4$ radians and when $x = 0,\ \theta = \tan^{-1}(0) = 0.$

Thus, when u varies from 1 to 0, θ varies from $\frac{\pi}{4}$ to 0. Therefore,

$$\int_{x=0}^{\frac{\pi}{2}} \frac{3\sin x}{1+\cos^2 x}\,dx = -3\int_{u=1}^{0}\frac{du}{1+u^2} = -3\int_{\theta=\pi/4}^{0}\frac{\sec^2\theta d\theta}{1+\tan^2\theta} = -3\int_{\theta=\pi/4}^{0}\frac{\sec^2\theta d\theta}{\sec^2\theta}$$

$$= -3\int_{\theta=\frac{\pi}{4}}^{0} 1\ d\theta = -3\int_{\theta=\frac{\pi}{4}}^{0}\theta^0 d\theta = -3\left[\frac{\theta^{0+1}}{0+1}\right]_{\theta=\frac{\pi}{4}}^{0} = -3\left[\frac{\theta}{1}\right]_{\theta=\frac{\pi}{4}}^{0}$$

$$= -3[\theta]_{\theta=\frac{\pi}{4}}^{0} = -3\left(0-\frac{\pi}{4}\right) = -3\left(-\frac{\pi}{4}\right) = \frac{3\pi}{4} \approx 2.356$$

Explanation:

- $\frac{\pi}{2}$ radians $= \frac{180^0}{2} = 90^0,\ \cos\frac{\pi}{2} = 0$ and $\cos 0 = 1.$
- $\frac{\pi}{4}$ radians $= \frac{180^0}{4} = 45^0,\ \tan\frac{\pi}{4} = 1$ or $\tan^{-1}(1) = \frac{\pi}{4}.$
- $\tan 0 = 0$ or $\tan^{-1}(0) = 0.$

- Recall that we have the identity $1 + \tan^2\theta = \sec^2\theta$.
- Note that $1 = \theta^0$ and $\theta^1 = \theta$.

8

$$\int_{\theta=0}^{\pi/2} \sqrt{\sin\theta}\cos^3\theta\, d\theta = \int_{\theta=0}^{\pi/2} \sqrt{\sin\theta}\cos^2\theta\cos\theta\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sqrt{\sin\theta}(1 - \sin^2\theta)\cos\theta\, d\theta$$

Let $u = \sin\theta$. Then, $\frac{du}{d\theta} = \cos\theta$, which gives $du = \cos\theta d\theta$ or $d\theta = du/\cos\theta$

Note that, when $\theta = 0, u = \sin 0 = 0$ and when $\theta = \pi/2, u = \sin\frac{\pi}{2} = 1$.

Thus, when θ varies from 0 to $\pi/2$, u varies from 0 to 1 and

$$\int_{\theta=0}^{\pi/2} \sqrt{\sin\theta}(1 - \sin^2\theta)\cos\theta\, d\theta = \int_{u=0}^{1} \sqrt{u}(1 - u^2)\cos\theta\left(\frac{du}{\cos\theta}\right)$$

$$= \int_{u=0}^{1} \sqrt{u}(1 - u^2)du = \int_{u=0}^{1} u^{1/2}(1 - u^2)du = \int_{u=0}^{1} (u^{1/2} - u^{1/2}u^2)du$$

$$= \int_{u=0}^{1} u^{1/2}du - \int_{u=0}^{1} u^{5/2}du = \left[\frac{u^{1/2+1}}{\frac{1}{2}+1}\right]_{u=0}^{1} - \left[\frac{u^{5/2+1}}{\frac{5}{2}+1}\right]_{u=0}^{1}$$

$$= \left[\frac{u^{3/2}}{\frac{3}{2}}\right]_0^1 - \left[\frac{u^{7/2}}{\frac{7}{2}}\right]_{u=0}^{1} = \frac{2}{3}[u^{3/2}]_{u=0}^{1} - \frac{2}{7}[u^{7/2}]_{u=0}^{1}$$

$$= \frac{2}{3}[(1)^{3/2} - (0)^{3/2}] - \frac{2}{7}\left[(1)^{\frac{7}{2}} - (0)^{\frac{7}{2}}\right] = \frac{2}{3}(1 - 0) - \frac{2}{7}(1 - 0) = \frac{2}{3}(1) - \frac{2}{7}(1) = \frac{2}{3} - \frac{2}{7} = \frac{14 - 6}{21}$$

$$= \frac{8}{21} \approx 0.381$$

Explanation:

- $\cos^2\theta = 1 - \sin^2\theta$ because we have the identity $\sin^2\theta + \cos^2\theta = 1$.
- $\frac{\pi}{2}$ radians $= \frac{180^0}{2} = 90^0$, $\sin\frac{\pi}{2} = 1$ and $\sin 0 = 0$.
- Since $x^m x^n = x^{m+n}$, we have $u^{1/2}u^2 = x^{1/2+2}$.
- Since fractions can be added only when they have common denominator, we have

$\frac{1}{2}+2$

$=\frac{1}{2}+\frac{4}{2}=\frac{1+4}{2}=\frac{5}{2}$, $\frac{1}{2}+1=\frac{1}{2}+\frac{2}{2}=\frac{1+2}{2}=\frac{3}{2}$ and $\frac{5}{2}+1=\frac{5}{2}+\frac{2}{2}=\frac{5+2}{2}=\frac{7}{2}$

.

9

$$\int \cos^4\theta\,d\theta = \int \cos^2\theta\cos^2\theta d\theta = \int (1-\sin^2\theta)\cos^2\theta d\theta$$

$$= \int (\cos^2\theta - \sin^2\theta\cos^2\theta)\,d\theta = \int [\cos^2\theta - (\sin\theta\cos\theta)^2]\,d\theta$$

$$= \int \left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{\sin 2\theta}{2}\right)^2\right] d\theta = \int \left[\frac{1+\cos 2\theta}{2} - \frac{\sin^2 2\theta}{4}\right] d\theta$$

$$= \int \left[\frac{1+\cos 2\theta}{2} - \frac{1}{4}\left(\frac{1-\cos 4\theta}{2}\right)\right] d\theta = \int \left[\frac{4+4\cos 2\theta}{8} - \frac{1-\cos 4\theta}{8}\right] d\theta$$

$$= \frac{1}{8}\int (4+4\cos 2\theta - 1 + \cos 4\theta)\,d\theta = \frac{1}{8}\int (3+4\cos 2\theta + \cos 4\theta)\,d\theta$$

$$= \frac{3}{8}\int 1 d\theta + \frac{4}{8}\int \cos 2\theta\,d\theta + \frac{1}{8}\int \cos 4\theta\,d\theta$$

$$= \frac{3}{8}\int \theta^0 d\theta + \frac{4}{8}\int \cos 2\theta\,d\theta + \frac{1}{8}\int \cos 4\theta\,d\theta$$

$$= \frac{3}{8}\left(\frac{\theta^{0+1}}{0+1}\right) + \frac{1}{2}\int \cos 2\theta\,d\theta + \frac{1}{8}\int \cos 4\theta\,d\theta$$

$$= \frac{3}{8}\left(\frac{\theta^1}{1}\right) + \frac{1}{2}\int \cos 2\theta\,d\theta + \frac{1}{8}\int \cos 4\theta\,d\theta$$

For evaluating $\int \cos 2\theta\,d\theta$, let $u = 2\theta$. Then, $\frac{du}{d\theta} = 2$, $du = 2d\theta$ or $d\theta = \frac{du}{2}$.

Again, for evaluating $\int \cos 4\theta\,d\theta$, let $v = 4\theta$.Then, $\frac{du}{d\theta} = 4$, $du = 4d\theta$ or $d\theta = \frac{du}{4}$.

Therefore,

$$\int \cos^4\theta\,d\theta = \frac{3}{8}\left(\frac{\theta^1}{1}\right) + \frac{1}{2}\int \cos u\left(\frac{du}{2}\right) + \frac{1}{8}\int \cos v\left(\frac{dv}{4}\right)$$

$$= \frac{3\theta}{8} + \frac{1}{4}\int \cos u du + \frac{1}{32}\int \cos v dv = \frac{3\theta}{8} + \frac{1}{4}\int \cos u du + \frac{1}{32}\int \cos v dv$$

$$= \frac{3\theta}{8} + \frac{1}{4}(\sin u) + \frac{1}{32}(\sin v) + c = \frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} + c$$

Explanation:

- Recall that $\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$ since $\sin 2\theta = 2\sin\theta\cos\theta$.
- Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we get $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$.
    - Using the identity $\cos 2\theta = 1 - 2\sin^2\theta$ and putting $\theta = 2\theta$, we get

$\cos 4\theta = 1 - 2\sin^2 2\theta$or $\sin^2 2\theta = \frac{1 - \cos 4\theta}{2}$. Note also that $1 = \theta^0$ and $\theta^1 = \theta$.

20 INTEGRATION BY PARTS

We wish to evaluate the integral $\int f(x)\ dx$ by making use of the following formula for integration by parts:

$$\int u\ dv = uv - \int v\ du$$

This formula is very useful when $f(x)$ is product of two functions like, for example, $f(x) = x \cos x$ or $f(x) = x^2 \ln x$.

For using this formula, follow these steps.

1. First step is to write $f(x)\ dx$ in the form $u\ dv$, where dv contains dx. For this we have to choose u and dv keeping in mind the following:

   (i) The integration by parts formula involves calculation of $\int v\ du$ on the right hand side. Therefore, u and dv should be so chosen that $\int v\ du$ becomes easier to calculate than $\int u\ dv$.

   (ii) It should be easy to differentiate u since going from u to du involves taking the derivative.

   (iii) It should be easy to calculate the antiderivative of v, since going from dv to v involves evaluation of the same.
2. Having chosen u, find $\frac{du}{dx}$, from which we can find du.
3. Evaluate the antiderivative of dv to get v.
4. Now, make use of the formula for integration by parts by substituting the values of u, v, dv and du obtained in steps 1 to 3.
5. Sometimes, our choice of u and dv might not result in $\int v\ du$ becoming easier to calculate than $\int u\ dv$. In such cases, we need to make a different choice of u and dv and again carry out the above steps till we get the desired result.

Experience of solving a variety of problems allows us to learn how to make right choice in the first instance.

For evaluation of definite integral, we make use the following formula for definite integrals

$$\int_a^b u\ dv = [uv]_a^b - \int_a^b v\ du$$

Example 1: Evaluate the following integral

$$\int x\cos x\ dx$$

Solution: While writing $\int f(x)\ dx$ in the form $\int u\ dv$, we should choose u and dv in such a way that that $\int v\ du$ becomes easier to perform than $\int u\ dv$. Now, going from u to du involves taking the derivative. In the present case, it seems that the derivative of x rather than the antiderivative of x is likely to simplify such integral. Also, it is easy to find the antiderivative of $\cos x$.

Therefore, let us try $u = x$. Then $dv = \cos x dx$.

When $u = x$, $\frac{du}{dx} = 1$ which gives $du = dx$ and

$$v = \int dv = \int \cos v dx = \sin x$$

Applying the integration by parts formula, we have

$$\int u\ dv = uv - \int v\ du$$

Substitute $u = x$, $v = \sin x$, $dv = \cos x dx$ and $du = dx$ in the above formula:

$$\int x\cos x\ dx = (x)(\sin x) - \int \sin x(dx)$$

$$= x\sin x - \int \sin x dx = x\sin x - (-\cos x) + c = x\sin x + \cos x + c$$

Check: Let $y = x\sin x + \cos x + c,$ then

$$\frac{dy}{dx} = \frac{d}{dx}(x\sin x + \cos x + c) = \frac{d}{dx}(x\sin x) + \frac{d}{dx}(\cos x) + \frac{d}{dx}(c)$$

We will apply the product rule to find $\frac{d}{dx}(x\sin x)$. For this, let

$$f(x) = x,\ g(x) = \sin x, \text{ so that } \frac{d}{dx}(x\sin x) = \frac{d}{dx}(fg) = \ ?$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(fg)\right] + \frac{d}{du}(\cos x) + \frac{d}{du}(c) = \left[f\frac{dg}{dx} + g\frac{df}{dx}\right] + (-\sin x) + 0$$

$$= \left[x\frac{d}{dx}(\sin x) + \sin x\frac{d(x)}{dx}\right] - \sin x = [x(\cos x) + \sin x(1)] - \sin x$$

$$= [x\cos x + \sin x] - \sin x = x\cos x + \sin x - \sin x$$

$$= x\cos x$$

Example 2: Evaluate the following integral

$$\int_{x=1}^{e} x^2 \ln x\ dx$$

Solution: First step is to write $\int f(x)\ dx$ in the form $\int u\ dv$. While doing this, we should choose u and dv in such a way that that $\int v\ du$ becomes easier to perform than $\int u\ dv$. Now, going from u to du involves taking the derivative. In the present example, it seems that the derivative of $\ln x$ rather than the antiderivative of x is both easier to calculate and also likely to simplify such integral. Moreover, it is easy to find the antiderivative of x^2.

Therefore, let us try $u = \ln x$. Then $dv = x^2 dx$.

When $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$ which gives $du = \frac{dx}{x}$ and

$$v = \int dv = \int x^2 dx = \frac{x^3}{3}$$

Applying the integration by parts formula, we have

$$\int_a^b u\ dv = [uv]_a^b - \int_a^b v\ du$$

Substituting the values, we have

$$\int_{x=1}^{e} x^2 \ln x\ dx = \left[(\ln x)\left(\frac{x^3}{3}\right)\right]_{x=1}^{e} - \int_{x=1}^{e} \frac{x^3}{3}\left(\frac{dx}{x}\right) = \frac{1}{3}\left[(\ln x)(x^3)\right]_{x=1}^{e} - \frac{1}{3}\int_{x=1}^{e} x^2 dx$$

$$= \frac{1}{3}[x^3 \ln x]_{x=1}^{e} - \frac{1}{3}\left[\frac{x^3}{3}\right]_{x=1}^{e} = \frac{1}{3}[x^3 \ln x]_{x=1}^{e} - \frac{1}{9}[x^3]_{x=1}^{e}$$

$$= \frac{1}{3}(e^3 \ln e - 1^3 \ln 1) - \frac{1}{9}(e^3 - 1^3)$$

$$= \frac{1}{3}(e^3(1) - (1)(0)) - \frac{1}{9}(e^3 - 1) = \frac{1}{3}(e^3 - 0) - \frac{1}{9}(e^3 - 1)$$

$$= \frac{1}{3}(e^3) - \frac{1}{9}(e^3 - 1) = \frac{3e^3 - e^3 + 1}{9} = \frac{2e^3 + 1}{9} \approx 4.575$$

Example 3: Evaluate the following integral

$$\int \ln x\ dx$$

Solution: Recall that we have used the following result in Chapter 17.

$$\int \ln x\ dx = x\ln x - x + c$$

We will now prove the above result by using the method of integration by parts.

First step is to write $\int \ln x \; dx$ in the form $\int u \; dv$. We will take constant function 1 as one of the functions and write $\int \ln x \; dx = \int \ln x 1 \; dx$. We should choose u and dv in such a way that that $\int v \; du$ becomes easier to perform than $\int u \; dv$. Now, going from u to du involves taking the derivative. In the present example, it seems that the derivative of $\ln x$ rather than the antiderivative of $\ln x$ is both easier to calculate and also likely to simplify such integral. Moreover, it is easy to find the antiderivative of 1. Therefore, let us try $u = \ln x$. Then $dv = 1 \; dx$.

When $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$ which gives $du = \frac{dx}{x}$ and

$$v = \int dv = \int 1 dx = \int x^0 dx = x$$

Applying the integration by parts formula, we have

$$\int u \; dv = uv - \int v \; du$$

Substituting the values, we have

$$\int \ln x \; dx = \int \ln x 1 \; dx = (\ln x)(x) - \int x\left(\frac{dx}{x}\right) = x \ln x - \int dx$$

$$= x \ln x - \int x^0 dx = x \ln x - \int x^0 dx = x \ln x - x + c$$

Check: $\frac{dy}{dx} = \frac{d}{dx}(x \ln x - x + c) = \frac{d}{dx}(x \ln x) - \frac{d}{dx}(x) + \frac{d}{dx}(c)$

Apply the product rule to find $\frac{d}{dx}(x \ln x)$.

$$\text{Let } f(x) = x, \; g(x) = \ln x, \; \text{so that } \frac{d}{dx}(x \ln x) = \frac{d}{dx}(fg) = ?$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(fg)\right] - \frac{d}{du}(x) + \frac{d}{du}(c)$$

$$= \left[f\frac{dg}{dx} + g\frac{df}{dx}\right] - 1 + 0 = \left(x\frac{d}{dx}(\ln x) + \ln x\frac{d(x)}{dx}\right) - 1$$

$$= \left[x\left(\frac{1}{x}\right) + \ln x(1)\right] - 1 = [1 + \ln x] - 1 = 1 + \ln x - 1 = \ln x$$

Chapter 20 Exercises

Set A

Evaluate the following integrals.

You can write your answers in the space provided for each question and check with solutions given at the end of the book.

1

$\int x\sec^2 x \, dx$

2

$\int \theta^2 \sin\theta d\theta =$

Set B

3

$\int_{x=\pi/6}^{\pi/3} \cos x \cot x \, dx =$

Set C

4

$\int x^2 e^x dx =$

5

$\int e^x \cos x \, dx =$

Set D

6

$\int_{x=0}^{1} xe^x \, dx =$

7

$\int_{x=1}^{2} \frac{\ln x}{x^2} \, dx =$

Chapter 20 Solutions

Set A

We will use integration by parts to evaluate $\int x\sec^2 x\, dx$.

Let $u = x$ and $dv = \sec^2 x dx$. Then, $\frac{du}{dx} = 1$, which gives $du = dx$ and

$$v = \int dv = \int \sec^2 x dx = \tan x$$

Applying the integration by parts formula, we have

$$\int u\ dv = uv - \int v\ du$$

$$\int x\sec^2 x\ dx = (x)(\tan x) - \int \tan x(dx) = x\tan x - \int \tan x dx$$

$$= x\tan x - (\ln|\sec x|) + c = x\tan x - \ln|\sec x| + c$$

2

We will use integration by parts to evaluate $\int \theta^2 \sin\theta d\theta$.

Let $u = \theta^2$ and $dv = \sin\theta d\theta$. Then, $\frac{du}{d\theta} = 2\theta$, which gives $du = 2\theta\ d\theta$ and

$$v = \int dv = \int \sin\theta d\theta = -\cos\theta$$

Applying the integration by parts formula, we have

$$\int u\ dv = uv - \int v\ du$$

$$\int \theta^2 \sin\theta d\theta = \theta^2(-\cos\theta) - \int(-\cos\theta)(2\theta\ d\theta) = -\theta^2\cos\theta + 2\int \theta\cos\theta d\theta$$

To evaluate $\int \theta\cos\theta d\theta$, we again apply integration by parts.

Let $w = \theta$ and $dz = \cos\theta d\theta$. Then, $\frac{dw}{d\theta} = 1$, which gives $dw = d\theta$ and

$$z = \int dz = \int \cos\theta d\theta = \sin\theta$$

Therefore, $\int \theta\cos\theta d\theta = \int w\ dz = wz - \int z dw$ and

$$\int \theta^2 \sin\theta d\theta = -\theta^2\cos\theta + 2\left(wz - \int z dw\right)$$

$$= -\theta^2\cos\theta + 2\left(\theta\sin\theta - \int \sin\theta\ d\theta\right) = -\theta^2\cos\theta + 2\theta\sin\theta - 2(-\cos\theta) + c$$

$$= -\theta^2\cos\theta + 2\theta\sin\theta + 2\cos\theta + c = (-\theta^2 + 2)\cos\theta + 2\theta\sin\theta + c$$

Chapter 20 – Set B

$$\int_{x=\pi/6}^{\pi/3} \cos x \cot x \, dx = ? \text{ Let } u = \cot x \text{ and } dv = \cos x \, dx.$$

Then, $\frac{du}{dx} = -\csc^2 x$, $du = \csc^2 x \, dx$ and

$v = \int dv = \int \cos x \, dx = \sin x$

Applying the integration by parts formula, we have $\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$

$$\int_{x=\pi/6}^{\pi/3} \cos x \cot x \, dx = [\cot x \sin x]_{x=\pi/6}^{\pi/3} - \int_{x=\pi/6}^{\pi/3} (\sin x)\left(-\csc^2 x\right) dx$$

$$= \left[\frac{\cos x}{\sin x}\sin x\right]_{x=\pi/6}^{\pi/3} - \int_{x=\pi/6}^{\pi/3} (\sin x)\left(-\frac{1}{\sin^2 x}\right) dx$$

$$= [\cos x]_{x=\pi/6}^{\pi/3} - \int_{x=\pi/6}^{\pi/3} \left(-\frac{1}{\sin x}\right) dx = [\cos x]_{x=\pi/6}^{\pi/3} + \int_{x=\pi/6}^{\pi/3} \csc x \, dx$$

$$= (\cos \pi/3 - \cos \pi/6) + [-\ln|\csc x + \cot x|]_{x=\pi/6}^{\pi/3}$$

$$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) - [\ln|\csc x + \cot x|]_{x=\pi/6}^{\pi/3}$$

$$= \frac{1-\sqrt{3}}{2} - \left[\ln\left|\csc\frac{\pi}{3} + \cot\frac{\pi}{3}\right| - \ln\left|\csc\frac{\pi}{6} + \cot\frac{\pi}{6}\right|\right]$$

$$= \frac{1-\sqrt{3}}{2} - \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| + \ln|2+\sqrt{3}| = \frac{1-\sqrt{3}}{2} - \ln\left(\frac{3}{\sqrt{3}}\right) + \ln(2+\sqrt{3})$$

$$= \frac{1-\sqrt{3}}{2} - \ln\sqrt{3} + \ln(2+\sqrt{3}) = \frac{1-\sqrt{3}}{2} + \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) \approx 0.402$$

Explanation:

- $\frac{\pi}{3}$ radians $= \frac{180^0}{3} = 60^0$, $\frac{\pi}{6}$ radians $= \frac{180^0}{3} = 60^0$.
- $\cos\frac{\pi}{3} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\csc\frac{\pi}{3} = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\sin(60^0)} = \frac{1}{\frac{\sqrt{3}}{2}} = 1 \times \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$, $\cot\frac{\pi}{3} = \frac{1}{\tan\frac{\pi}{3}}$
- $= \frac{1}{\sqrt{3}}$, $\csc\frac{\pi}{6} = \frac{1}{\sin\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{1}{2}} = 1 \times \frac{2}{1} = 2$, $\cot\left(\frac{\pi}{6}\right) = \frac{1}{\tan\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{1}{\sqrt{3}}} = 1 \times \frac{\sqrt{3}}{1} = \sqrt{3}$.

Chapter 20 – Set C

[redacted] We will use integration by parts to evaluate $\int x^2 e^x dx$.

Let $u = x^2$ and $dv = e^x\,dx$. Then, $\frac{du}{dx} = 2x$, which gives $du = 2x\,dx$ and

$v = \int dv = \int e^x\,dx = e^x$

Applying the integration by parts formula, we have

$$\int u\,dv = uv - \int v\,du$$

$$\int x^2e^x dx = (x^2)(e^x) - \int(e^x)(2x\,dx) = x^2e^x - 2\int xe^x\,dx$$

To evaluate $\int xe^x\,dx$, we again integrate by parts.

Let $w = x$ and $dz = e^x\,dx$, $\frac{dw}{d\theta} = 1$, which gives $dw = dx$ and

$z = \int dz = \int e^x\,dx = e^x$

$$\int w\,dz = wz - \int z\,dw$$

Therefore,

$$\int x^2e^x dx = x^2e^x - 2\int w\,dz = x^2e^x - 2\left(wz - \int z\,dw\right)$$

$$= x^2e^x - 2\left(xe^x - \int e^x\,dx\right) = x^2e^x - 2xe^x + 2\int e^x\,dx$$

$$= x^2e^x - 2xe^x + 2e^x + c = e^xx^2 - 2e^xx + 2e^x + c = e^x\left(x^2 - 2x + 2\right) + c$$

Check: Apply the product rule. Let $f(x) = e^x$, $\quad g(x) = x^2 - 2x + 1,$

$$\text{so that } \frac{d}{dx}\left[e^x(x^2 - 2x + 2) + c\right] = \frac{d}{dx}(fg) + \frac{d}{dx}c = \frac{d}{dx}(fg) + 0 = \frac{d}{dx}(fg)$$

$$= f\frac{dg}{dx} + g\frac{df}{dx} = e^x\frac{d}{dx}(x^2 - 2x + 2) + (x^2 - 2x + 2)\frac{d}{dx}(e^x)$$

$$= e^x(2x - 2 + 0) + (x^2 - 2x + 2)e^x = e^x(2x - 2) + e^x(x^2 - 2x + 2)$$

$$= e^x(2x - 2 + x^2 - 2x + 2) = e^xx^2 = x^2e^x$$

[redacted]

$$\int e^x\cos x\,dx = ?$$

We will use integration by parts to evaluate $\int e^x\cos x\,dx$.

Let $u = \cos x$ and $dv = e^x\,dx$ $\frac{du}{dx} = -\sin x$, which gives $du = -\sin x\,dx$ and

$v = \int dv = \int e^x\,dx = e^x$

$$\int u\ dv = uv - \int v\ du$$

$$\int e^{x}\cos x\ dx = (\cos x)(e^{x}) - \int e^{x}(-\sin x\ dx) = e^{x}\cos x + \int e^{x}\sin x dx$$

To evaluate $\int e^{x}\sin x dx$, we again integrate by parts.

Now, let $w = \sin x$ and $dz = e^{x}\ dx$. Then $\frac{dw}{dx} = \cos x,\ \ dw = \cos x\ dx$ and

$$z = \int dz = \int e^{x}\ dx = e^{x}$$

$$\int e^{x}\cos x\ dx = e^{x}\cos x + \left[wz - \int z\ dw\right] = e^{x}\cos x + \left[\sin x e^{x} - \int e^{x}\cos x\ dx\right]$$

$$= e^{x}\cos x + e^{x}\sin x - \int e^{x}\cos x\ dx = e^{x}(\cos x + \sin x) - \int e^{x}\cos x\ dx$$

Now, adding $\int e^{x}\cos x\ dx$ to both sides of the above equation, we get

$$\int e^{x}\cos x\ dx + \int e^{x}\cos x\ dx = e^{x}(\cos x + \sin x) - \int e^{x}\cos x\ dx + \int e^{x}\cos x\ dx$$

$$2\int e^{x}\cos x\ dx = e^{x}(\cos x + \sin x)$$

$$\text{Hence, } \int e^{x}\cos x\ dx = \frac{e^{x}}{2}(\cos x + \sin x) + c$$

Check: Apply the product rule. Let $f(x) = \frac{e^{x}}{2},\quad g(x) = \cos x + \sin x,$

$$\text{so that } \frac{d}{dx}\left[\frac{e^{x}}{2}(\cos x + \sin x) + c\right] = \frac{d}{dx}(fg) + \frac{d}{dx}c = \frac{d}{dx}(fg) + 0 = \frac{d}{dx}(fg)$$

$$= f\frac{dg}{dx} + g\frac{df}{dx} = \frac{e^{x}}{2}\frac{d}{dx}(\cos x + \sin x) + (\cos x + \sin x)\frac{d}{dx}\left(\frac{e^{x}}{2}\right)$$

$$= \frac{e^{x}}{2}\frac{d}{dx}(\cos x) + \frac{e^{x}}{2}\frac{d}{dx}(\sin x) + (\cos x + \sin x)\frac{d}{dx}\left(\frac{e^{x}}{2}\right)$$

$$= \frac{e^{x}}{2}(-\sin x) + \frac{e^{x}}{2}(\cos x) + (\cos x + \sin x)\left(\frac{e^{x}}{2}\right)$$

$$= -\frac{e^{x}}{2}\sin x + \frac{e^{x}}{2}\cos x + \frac{e^{x}}{2}\cos x + \frac{e^{x}}{2}\sin x = \frac{e^{x}}{2}\cos x + \frac{e^{x}}{2}\cos x$$

$$= 2\left(\frac{e^{x}}{2}\cos x\right) = e^{x}\cos x$$

Chapter 20 – Set D

[redacted]

$$\int_{x=0}^{1} xe^{x}\,dx = ?$$

Let $u = x$ and $dv = e^{x}\,dx$. Then, $\frac{du}{dx} = 1$, $du = dx$ and $v = \int dv = \int e^{x}\,dx = e^{x}$.

Applying the integration by parts formula: $\int_a^b u\,dv$

$= [uv]_a^b - \int_a^b v\,du$, we have

$$\int_{x=0}^{1} xe^{x}\,dx = [xe^{x}]_{x=0}^{1} - \int_{x=0}^{1} e^{x}dx = [xe^{x}]_{x=0}^{1} - [e^{x}]_{x=0}^{1}$$

$$= \left[(1)(e^{1}) - (0)(e^{0})\right] - (e^{1} - e^{0}) = [(1)(e) - (0)(1)] - (e - 1)$$

$$= (e - 0) - (e - 1) = e - e + 1 = 1$$

7

$$\int_{x=1}^{2} \frac{\ln x}{x^{2}}\,dx = ?$$

Let $u = \ln x$ and $dv = \frac{dx}{x^{2}}$. Then, $\frac{du}{dx} = \frac{d}{dx}\ln x = \frac{1}{x}$, or $du = \frac{dx}{x}$ and

$$v = \int dv = \int \frac{dx}{x^{2}} = \int x^{-2}dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

Applying the integration by parts formula: $\int_a^b u\,dv$

$= [uv]_a^b - \int_a^b v\,du$, we have

$$\int_{x=1}^{2} \frac{\ln x}{x^{2}}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_{x=1}^{2} - \int_{x=1}^{2}\left(-\frac{1}{x}\right)\left(\frac{dx}{x}\right) = -\left[\frac{\ln x}{x}\right]_{x=1}^{2} + \int_{x=1}^{2}\frac{dx}{x^{2}}$$

$$= -\left[\frac{\ln x}{x}\right]_{x=1}^{2} + \int_{x=1}^{2} x^{-2}dx = -\left[\frac{\ln x}{x}\right]_{x=1}^{2} + \left[\frac{x^{-2+1}}{-2+1}\right]_{x=1}^{2}$$

$$= -\left[\frac{\ln x}{x}\right]_{x=1}^{2} + \left[\frac{x^{-1}}{-1}\right]_{x=1}^{2} = -\left[\frac{\ln x}{x}\right]_{x=1}^{2} - \left[\frac{1}{x}\right]_{x=1}^{2}$$

$$= -\left(\frac{\ln 2}{2} - \frac{\ln 1}{2}\right) - \left(\frac{1}{2} - \frac{1}{1}\right) = -\left(\frac{\ln 2}{2} - \frac{0}{1}\right) - \left(-\frac{1}{2}\right) = -\frac{\ln 2}{2} + \frac{1}{2} \approx 0.153$$

Explanation:

- Note that $\ln 1 = 0$ and $\frac{1}{2} - \frac{1}{1} = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$.

21 MULTIPLE INTEGRALS

In this chapter we will learn how to calculate double integrals and triple integrals that are definite integrals of functions of two and three variables respectively. We follow the following steps for evaluating such integrals:

1. First step is to look at the limits of integration to check if all variables have constant limits or if one or more variables have variable limits. If all variables have constant limits, we can perform the integrals in any order we wish. On the other hand, if, for example, integral over y has limits that depend on x, we must first carry out the integral over y before integrating over x While doing so, we may change the order of the differential elements $dx\ dy$ depending on the variable over which integral is being carried out first.
2. While integrating over one variable, we put the integral over this particular variable in parenthesis. Also, when integrating over a particular variable, we must treat the other independent variables as constants. For example, in case of a double integral, while carrying out integral over x, we treat the variable y as constant.
3. Having finished integral over one variable, evaluate the antiderivative over the limits of this variable before integrating over the next integral.

Example 1: Evaluate the following integral

$$\int_{x=1}^{3}\int_{y=0}^{1} xy\ \ dx\ dy$$

Solution: Note that integrals over both x and y have constant limits. Therefore, we can perform the integrals over these variables in any order we wish. In this case, we choose to first integrate over y and then integrate over x. We put the first integral over y in parenthesis. While carrying out integral over y, we treat the variable x as constant and can take x out of the parenthesis as shown below.

$$\int_{x=1}^{3}\int_{y=0}^{1} xy\ \ dx\ dy = \int_{x=1}^{3} x\left(\int_{y=0}^{1} y\ \ dy\right)dx = \int_{x=1}^{3} x\left[\frac{y^2}{2}\right]_0^1 dx = \int_{x=1}^{3} x\left(\frac{1^2}{2} - \frac{0^2}{2}\right)dx$$

$$= \int_{x=1}^{3} x\left(\frac{1}{2} - 0\right)dx = \int_{x=1}^{3} x\left(\frac{1}{2}\right)dx = \frac{1}{2}\int_{x=1}^{3} xdx = \frac{1}{2}\left[\frac{x^2}{2}\right]_1^3 = \frac{1}{2}\left(\frac{3^2}{2} - \frac{1^2}{2}\right)$$

$$= \frac{1}{2}\left(\frac{9}{2} - \frac{1}{2}\right) = \frac{1}{2}\left(\frac{8}{2}\right) = \frac{1}{2}(4) = 2$$

Example 2: Evaluate the following integral

$$\int_{x=0}^{2} \int_{y=1}^{x} \frac{\sqrt{x}}{y^2}\, dx\, dy$$

Solution: We observe that the integral over y has limits that depend on x. Therefore, we must first carry out the integral over y before integrating over x. We put the first integral over y in parenthesis. While carrying out integral over y, we treat the variable x as constant. Therefore, we can take $\sqrt{x}$ out of the parenthesis as shown below.

$$\int_{x=0}^{2} \int_{y=1}^{x} \frac{\sqrt{x}}{y^2}\, dx\, dy = \int_{x=0}^{2} \sqrt{x}\left(\int_{y=1}^{x} y^{-2}\, dy \right) dx = \int_{x=0}^{2} \sqrt{x}\left[\frac{y^{-2+1}}{-2+1} \right]_1^x dx$$

$$= \int_{x=0}^{2} \sqrt{x}\left[\frac{y^{-1}}{-1} \right]_1^x dx = -\int_{x=0}^{2} \sqrt{x}\left[y^{-1} \right]_1^x dx = -\int_{x=0}^{2} \sqrt{x}\left(x^{-1} - (1)^{-1} \right) dx$$

$$= -\int_{x=0}^{2} \sqrt{x}\left(x^{-1} - 1 \right) dx = -\int_{x=0}^{2} x^{1/2}\left(x^{-1} - 1 \right) dx = -\int_{x=0}^{2} \left(x^{-1/2} - x^{1/2} \right) dx$$

$$= -\int_{x=0}^{2} x^{-1/2} dx - \int_{x=0}^{2} \left(-x^{1/2} \right) dx = -\int_{x=0}^{2} x^{-1/2} dx + \int_{x=0}^{2} x^{1/2} dx$$

$$= -\left[\frac{x^{-1/2+1}}{-\frac{1}{2}+1} \right]_0^2 + \left[\frac{x^{1/2+1}}{\frac{1}{2}+1} \right]_0^2 = -\left[\frac{x^{1/2}}{\frac{1}{2}} \right]_0^2 + \left[\frac{x^{3/2}}{\frac{3}{2}} \right]_0^2$$

$$= -2\left[x^{1/2} \right]_0^2 + \left(\frac{2}{3} \right)\left[x^{3/2} \right]_0^2 = -2\left[(2)^{1/2} - (0)^{1/2} \right] + \left(\frac{2}{3} \right)\left[(2)^{3/2} - (0)^{3/2} \right]$$

$$= -2\left[(2)^{1/2} \right] + \left(\frac{2}{3} \right)\left[(2)^{3/2} \right] = -2(2)^{1/2} + \left(\frac{2}{3} \right)(2)^{3/2} = -2(2)^{1/2} + \left(\frac{2}{3} \right)\left[2(2)^{1/2} \right]$$

$$= 2(2)^{1/2}\left[-1 + \frac{2}{3} \right] = 2(2)^{1/2}\left[-\frac{1}{3} \right] = -\frac{2(2)^{\frac{1}{2}}}{3} = -\frac{2\sqrt{2}}{3} \approx -0.943$$

Example 3: Evaluate the following integral

$$\int_{x=0}^{y}\int_{y=1}^{3}\sqrt{\frac{y}{x}}dx\,dy$$

Solution: We observe that the integral over has x limits that depend on y. Therefore, we must first carry out the integral over x before integrating over y. We put the first integral over x in parenthesis. While carrying out integral over x, we treat the variable y as constant. Therefore, we can take $\sqrt{y}$ out of the parenthesis as shown below.

$$\int_{x=0}^{y}\int_{y=1}^{3}\sqrt{\frac{y}{x}}dx\,dy = \int_{x=0}^{y}\int_{y=1}^{3}\sqrt{\frac{y}{x}}dx\,dy = \int_{x=0}^{y}\int_{y=1}^{3}\frac{\sqrt{y}}{\sqrt{x}}dx\,dy$$

$$= \int_{y=1}^{3}\sqrt{y}\left(\int_{x=0}^{y}\frac{1}{\sqrt{x}}dx\right)dy = \int_{y=1}^{3}\sqrt{y}\left(\int_{x=0}^{y}\frac{1}{x^{1/2}}dx\right)dy = \int_{y=1}^{3}\sqrt{y}\left(\int_{x=0}^{y}x^{-1/2}dx\right)dy$$

$$= \int_{y=1}^{3}\sqrt{y}\left[\frac{x^{-1/2+1}}{-\frac{1}{2}+1}\right]_0^y dy = \int_{y=1}^{3}\sqrt{y}\left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^y dy = \int_{y=1}^{3}\sqrt{y}\,[2x^{1/2}]_0^y dy$$

$$= 2\int_{y=1}^{3}\sqrt{y}\,[x^{1/2}]_0^y dy = 2\int_{y=1}^{3}\sqrt{y}\,(y^{1/2} - 0^{1/2})dy = 2\int_{y=1}^{3}y^{1/2}\,(y^{1/2} - 0)dy$$

$$= 2\int_{y=1}^{3}y^{1/2}(y^{1/2})dy = 2\int_{y=1}^{3}y\,dy = 2\left[\frac{y^2}{2}\right]_1^3 = \frac{2}{2}[y^2]_1^3 = [y^2]_1^3$$

$$= (3^2 - 1^2) = (9 - 1) = 8$$

Example 4: Evaluate the following integral

$$\int_{x=y}^{2y}\int_{y=0}^{z}\int_{z=0}^{3}\frac{xy}{z^2}\,dx\,dy\,dz$$

Solution: Note that the integrals over x and y have variable limits whereas the integral over z has constant limits. Therefore, we must first carry out the integrals over x and y before integrating over z. Also, since the integral over has x limits that depend on y and the integral

over y has limits that depend on z, we should integrate over x before that over y. Hence, the order in which integrations are to be carried out is as follows. First, integrate over x, then over y and finally over z. While carrying out integral over x, for example, we treat y and z as constants and take y and $\frac{1}{z^2}$ out of the parenthesis.

$$\int_{x=y}^{2y}\int_{y=0}^{z}\int_{z=0}^{3}\frac{xy}{z^2}\,dx\,dydz = \int_{x=y}^{2y}\int_{y=0}^{z}\int_{z=0}^{3}\frac{xy}{z^2}\,dx\,dydz$$

$$= \int_{z=0}^{3}\frac{1}{z^2}\int_{y=0}^{z} y\left(\int_{x=y}^{2y} x\,dx\right)dy\,dz = \int_{z=0}^{3}\frac{1}{z^2}\int_{y=0}^{z} y\left[\frac{x^2}{2}\right]_{y}^{2y}dy\,dz$$

$$= \frac{1}{2}\int_{z=0}^{3}\frac{1}{z^2}\int_{y=0}^{z} y\,\left[x^2\right]_{y}^{2y}dy\,dz = \frac{1}{2}\int_{z=0}^{3}\frac{1}{z^2}\int_{y=0}^{z} y\,\left((2y)^2 - y^2\right)dy\,dz$$

$$= \frac{1}{2}\int_{z=0}^{3}\frac{1}{z^2}\int_{y=0}^{z} y\,\left((2y)^2 - y^2\right)dy\,dz = \frac{1}{2}\int_{z=0}^{3}\frac{1}{z^2}\int_{y=0}^{z} y\,\left(4y^2 - y^2\right)dy\,dz$$

$$= \frac{1}{2}\int_{z=0}^{3}\frac{1}{z^2}\int_{y=0}^{z} y\,\left(3y^2\right)dy\,dz = \frac{3}{2}\int_{z=0}^{3}\frac{1}{z^2}\int_{y=0}^{z} y^3\,dy\,dz$$

$$= \frac{3}{2}\int_{z=0}^{3}\frac{1}{z^2}\left(\int_{y=0}^{z} y^3\,dy\right)dz = \frac{3}{2}\int_{z=0}^{3}\frac{1}{z^2}\left[\frac{y^4}{4}\right]_0^z dz = \frac{3}{8}\int_{z=0}^{3}\frac{1}{z^2}\left(z^4 - 0^4\right)dz$$

$$= \frac{3}{8}\int_{z=0}^{3}\frac{1}{z^2}\left(z^4 - 0\right)dz = \frac{3}{8}\int_{z=0}^{3}\frac{z^4}{z^2}\,dz = \frac{3}{8}\int_{z=0}^{3} z^2\,dz = \frac{3}{8}\left[\frac{z^3}{3}\right]_0^3 = \frac{1}{8}\left[z^3\right]_0^3$$

$$= \frac{1}{8}\left(3^3 - 0^3\right) = \frac{1}{8}(27 - 0) = \frac{27}{8} \approx 3.375$$

Chapter 21 Exercises

Set A

Evaluate the following integrals.

You can write your answers in a notebook and check with solutions given at the end of this chapter.

1

$$\int_{x=0}^{2} \int_{y=1}^{3} \sqrt{x}\,dx\,dy =$$

2

$$\int_{x=0}^{2} \int_{y=1}^{3} \int_{z=0}^{3} x^2 y^2 z \, dx \, dy \, dz =$$

Set B

3

$$\int_{x=0}^{3} \int_{y=1}^{x} x \, dx \, dy =$$

4

$$\int_{x=1}^{3} \int_{y=0}^{x^2} \frac{y^2}{x} dx \, dy =$$

Set C

5

$$\int_{x=0}^{2} \int_{y=0}^{z} \int_{z=0}^{\sqrt{x}} xy \, dx \, dydz =$$

6

$$\int_{x=0}^{1/y} \int_{y=1}^{z} \int_{z=0}^{2} x^2 y^3 \, z \, dx \, dy \, dz \; =$$

Chapter 21 Solutions

Set A

1

$$\int_{x=0}^{2}\int_{y=1}^{3}\sqrt{x}\,dx\,dy = \int_{x=0}^{2}\sqrt{x}\left(\int_{y=1}^{3} 1\ dy\right)dx = = \int_{x=0}^{2}\sqrt{x}\left(\int_{y=1}^{3} y^0\ \ dy\right)dx$$

$$= \int_{x=0}^{2}\sqrt{x}\left[\frac{y^1}{1}\right]_{y=1}^{3}dx = \int_{x=0}^{2}\sqrt{x}[y]_{y=1}^{3}dx = \int_{x=0}^{2}\sqrt{x}(3-1)dx = \int_{x=0}^{2}\sqrt{x}(2)dx$$

$$= 2\int_{x=0}^{2}\sqrt{x}dx = 2\int_{x=0}^{2}x^{\frac{1}{2}}dx = 2\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{x=0}^{2} = 2\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_{x=0}^{2} = 2\left(\frac{2}{3}\right)[x^{3/2}]_{x=0}^{2} =$$

$$\frac{4}{3}((2)^{3/2} - (0)^{3/2}) = \frac{4}{3}(2\sqrt{2} - 0) = \frac{4}{3}(2\sqrt{2}) = \frac{8\sqrt{2}}{3} \approx 3.771$$

Explanation: Note that $1 = y^0$ and $y^1 = y$ and $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$.

- Since $x^{m+n} = x^m x^n$, we have $2^{3/2} = 2^{1+1/2} = 2^1 2^{1/2} = 2\sqrt{2}$.

2

$$\int_{x=0}^{2}\int_{y=1}^{3}\int_{z=0}^{3} x^2y^2z\ dx\ dy\ dz = \int_{x=0}^{2} x^2 \int_{y=1}^{3} y^2\left(\int_{z=0}^{3} z\ \ dz\right)dy\ dx$$

$$= \int_{x=0}^{2} x^2 \int_{y=1}^{3} y^2\left[\frac{z^2}{2}\right]_{z=0}^{3}dy\ dx = \frac{1}{2}\int_{x=0}^{2} x^2 \int_{y=1}^{3} y^2[z^2]_{z=0}^{3}dy\ dx$$

$$= \frac{1}{2}\int_{x=0}^{2} x^2 \int_{y=1}^{3} y^2(3^2 - 0^2)dy\ dx = \frac{9}{2}\int_{x=0}^{2} x^2\left(\int_{y=1}^{3} y^2\ dy\right)dx$$

$$= \frac{9}{2}\int_{x=0}^{2} x^2\left[\frac{y^3}{3}\right]_{y=1}^{3} dx = \left(\frac{9}{2}\right)\left(\frac{1}{3}\right)\int_{x=0}^{2} x^2[y^3]_{y=1}^{3} dx = \frac{3}{2}\int_{x=0}^{2} x^2(3^3 - 1^3)\ dx$$

$$= \frac{3}{2}\int_{x=0}^{2} x^2(27-1)\ dx = \left(\frac{3}{2}\right)(26)\int_{x=0}^{2} x^2\ dx = 39\int_{x=0}^{2} x^2\ dx = 39\left[\frac{x^3}{3}\right]_{x=0}^{2}$$

$$= \left(\frac{39}{3}\right)[x^3]_{x=0}^{2} = 13(2^3 - 0^3) = 13(8-0) = (13)(8) = 104$$

Set B

3

$$\int_{x=0}^{3}\int_{y=1}^{x} x\,dx\,dy = \int_{x=0}^{3} x\left(\int_{y=1}^{x} 1dy\right)dx = \int_{x=0}^{3} x\left(\int_{y=1}^{x} y^0\ dy\right)dx$$

$$= \int_{x=0}^{3} x\left[\frac{y^{0+1}}{0+1}\right]_{y=1}^{x} dx = \int_{x=0}^{3} x\left[\frac{y^1}{1}\right]_{y=1}^{x} dx = \int_{x=0}^{3} x\,[y]_{y=1}^{x} dx$$

$$= \int_{x=0}^{3} x\,(x-1)dx = \int_{x=0}^{3} (x^2 - x)dx = \int_{x=0}^{3} x^2dx - \int_{x=0}^{3} xdx$$

$$= \left[\frac{x^{2+1}}{2+1}\right]_{x=0}^{3} - \int_{x=0}^{3} x^1dx = \left[\frac{x^3}{3}\right]_{x=0}^{3} - \left[\frac{x^{1+1}}{1+1}\right]_{x=0}^{3} = \frac{1}{3}[x^3]_{x=0}^{3} - \left[\frac{x^2}{2}\right]_{x=0}^{3}$$

$$= \frac{1}{3}[x^3]_{x=0}^{3} - \frac{1}{2}[x^2]_{x=0}^{3}$$

$$= \frac{1}{3}(3^3 - 0^3) - \frac{1}{2}(3^2 - 0) = \frac{1}{3}(27) - \frac{1}{2}(9) = 9 - \frac{9}{2} = \frac{18}{2} - \frac{9}{2} = \frac{9}{2} = 4.5$$

Explanation: Note that $1 = y^0 = x^0$ and $x^1 = x,\ y^1 = y$.

4

$$\int_{x=1}^{3}\int_{y=0}^{x^2} \frac{y^2}{x}dx\,dy = \int_{x=1}^{3} \frac{1}{x}\left(\int_{y=0}^{x^2} y^2dy\right)dx = \int_{x=1}^{3} \frac{1}{x}\left[\frac{y^{2+1}}{2+1}\right]_{y=0}^{x^2} dx$$

$$= \int_{x=1}^{3} \frac{1}{x}\left[\frac{y^3}{3}\right]_{y=0}^{x^2} dx = \frac{1}{3}\int_{x=1}^{3} \frac{1}{x}[y^3]_{y=0}^{x^2} dx = \frac{1}{3}\int_{x=1}^{3} \frac{1}{x}[(x^2)^3 - 0^3]dx$$

$$= \frac{1}{3}\int_{x=1}^{3} \frac{1}{x}(x^6 - 0)dx = \frac{1}{3}\int_{x=1}^{3} \frac{1}{x}(x^6)dx = \frac{1}{3}\int_{x=1}^{3} x^5dx = \frac{1}{3}\left[\frac{x^{5+1}}{5+1}\right]_{x=1}^{3}$$

$$= \frac{1}{3}\left[\frac{x^6}{6}\right]_{x=1}^{3} = \left(\frac{1}{3}\right)\left(\frac{1}{6}\right)[x^6]_1^3 = \frac{1}{18}(3^6 - 1^6) = \frac{1}{18}(729 - 1)$$

$$= \frac{1}{18}(728) = \frac{364}{9} \approx 40.444$$

Chapter 21 – Set C

5

$$\int_{x=0}^{2}\int_{y=0}^{z}\int_{z=0}^{\sqrt{x}} xy\,dx\,dydz = \int_{x=0}^{2} x\int_{z=0}^{\sqrt{x}}\left(\int_{y=0}^{z} y\,dy\right)dz\,dx$$

$$= \int_{x=0}^{2} x\int_{z=0}^{\sqrt{x}}\left[\frac{y^2}{2}\right]_{y=0}^{z}dz\,dx = \frac{1}{2}\int_{x=0}^{2} x\int_{z=0}^{\sqrt{x}}[y^2]_{y=0}^{z}dz\,dx$$

$$= \frac{1}{2}\int_{x=0}^{2} x\int_{z=0}^{\sqrt{x}}(z^2 - 0^2)dz\,dx = \frac{1}{2}\int_{x=0}^{2} x\left(\int_{z=0}^{\sqrt{x}} z^2\,dz\right)dx = \frac{1}{2}\int_{x=0}^{2} x\left[\frac{z^3}{3}\right]_{z=0}^{\sqrt{x}}dx$$

$$= \frac{1}{6}\int_{x=0}^{2} x\,[z^3]_{z=0}^{\sqrt{x}}\,dx = \frac{1}{6}\int_{x=0}^{2} x\left[(\sqrt{x})^3 - 0^3\right]dx = \frac{1}{6}\int_{x=0}^{2} x\left[(x^{1/2})^3\right]dx$$

$$= \frac{1}{6}\int_{x=0}^{2} x\,(x^{3/2})\,dx = \frac{1}{6}\int_{x=0}^{2} x^{5/2}\,dx = \frac{1}{6}\left[\frac{x^{5/2+1}}{\frac{5}{2}+1}\right]_{x=0}^{2} = \frac{1}{6}\left[\frac{x^{7/2}}{\frac{7}{2}}\right]_{x=0}^{2}$$

$$= \frac{1}{6}\left(\frac{2}{7}\right)[x^{7/2}]_{x=0}^{2} = \frac{1}{21}[x^{7/2}]_{x=0}^{2} = \frac{1}{21}[2^{7/2} - (0)^{7/2}] = \frac{1}{21}(2^3 2^{1/2}) = \frac{1}{21}(8\sqrt{2}) = \frac{8\sqrt{2}}{21} \approx 0.539$$

Explanation:

- Note that $\sqrt{x} = x^{1/2}$ and $(x^{1/2})^3 = x^{3/2}$, since $(x^m)^n = x^{mn}$.
- $\frac{5}{2} + 1 = \frac{5}{2} + \frac{2}{2} = \frac{5+2}{2} = \frac{7}{2}$.
- Since $x^{m+n} = x^m x^n$, we have $2^{7/2} = 2^{3+1/2} = 2^3 2^{1/2} = 8\sqrt{2}$.

$$\int_{x=0}^{1/y}\int_{y=1}^{z}\int_{z=0}^{2} x^2y^3 z\,dx\,dy\,dz = \int_{z=0}^{2} z\,dz\int_{y=1}^{z} y^3\left(\int_{x=0}^{1/y} x^2\,dx\right)dy$$

$$= \int_{z=0}^{2} z\,dz\int_{y=1}^{z} y^3\left[\frac{x^{2+1}}{2+1}\right]_{x=0}^{1/y}dy = \int_{z=0}^{2} z\,dz\int_{y=1}^{z} y^3\left[\frac{x^3}{3}\right]_{x=0}^{1/y}dy$$

$$= \frac{1}{3}\int_{z=0}^{2} z\,dz \int_{y=1}^{z} y^3\,[x^3]_{x=0}^{1/y} dy = \frac{1}{3}\int_{z=0}^{2} z\,dz \int_{y=1}^{z} y^3 \left[\left(\frac{1}{y}\right)^3 - 0^3\right] dy$$

$$= \frac{1}{3}\int_{z=0}^{2} z\,dz \int_{y=1}^{z} y^3 \left[\frac{1}{y^3} - 0\right] dy = \frac{1}{3}\int_{z=0}^{2} z\,dz \int_{y=1}^{z} y^3 \frac{1}{y^3} dy = \frac{1}{3}\int_{z=0}^{2} z\,dz \int_{y=1}^{z} 1\,dy$$

$$= \frac{1}{3}\int_{z=0}^{2} z\,dz \int_{y=1}^{z} y^0\,dy = \frac{1}{3}\int_{z=0}^{2} z\,dz \left[\frac{y^{0+1}}{0+1}\right]_{y=1}^{z} = \frac{1}{3}\int_{z=0}^{2} z\,dz [y]_{y=1}^{z}$$

$$= \frac{1}{3}\int_{z=0}^{2} z\,dz[z-1] = \frac{1}{3}\int_{z=0}^{2} z^2 dz - \frac{1}{3}\int_{z=0}^{2} z dz = \frac{1}{3}\left[\frac{z^{2+1}}{2+1}\right]_{z=0}^{1} - \frac{1}{3}\left[\frac{z^{1+1}}{1+1}\right]_{z=0}^{1}$$

$$= \frac{1}{3}\left[\frac{z^3}{3}\right]_{z=0}^{2} - \frac{1}{3}\left[\frac{z^2}{2}\right]_{z=0}^{2} = \frac{1}{9}[z^3]_{z=0}^{2} - \frac{1}{6}[z^2]_{z=0}^{2}$$

$$= \frac{1}{9}(2^3 - 0^3) - \frac{1}{6}(2^2 - 0^2) = \frac{1}{9}(8 - 0) - \frac{1}{6}(4 - 0)$$

$$= \frac{1}{9}(8) - \frac{1}{6}(4) = \frac{8}{9} - \frac{4}{6} = \frac{8 \times 2}{9 \times 2} - \frac{4 \times 3}{6 \times 3} = \frac{16}{18} - \frac{12}{18}$$

$$= \frac{16 - 12}{18} = \frac{4}{18} = \frac{2}{9} \approx 0.222$$

TO THE READER

A Word of Thanks...

Thank you very much for going through the workbook. I sincerely hope that it has led you to realize that evaluation of derivatives, limits and integrals is not as difficult as it is often perceived to be!

...and a Request

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ABOUT THE AUTHOR

Sudhir K. Sood obtained his Ph.D. degree in fundamental particle physics. Subsequently, as research scientist and Professor of Physics at Universities in France, Canada and India, Dr. Sood has taught a number of courses both at introductory and advanced graduate level. He has lectured at international Physics conferences and authored numerous well-cited research papers that are published in reputed peer reviewed journals. More recently, for more than a decade, he has taught physics and mathematics to students who wish to specialize in engineering, medicine and physical science courses. During this period, Dr. Sood has also authored several physics books including one on calculus - based physics.

OTHER TITLES BY THE AUTHOR

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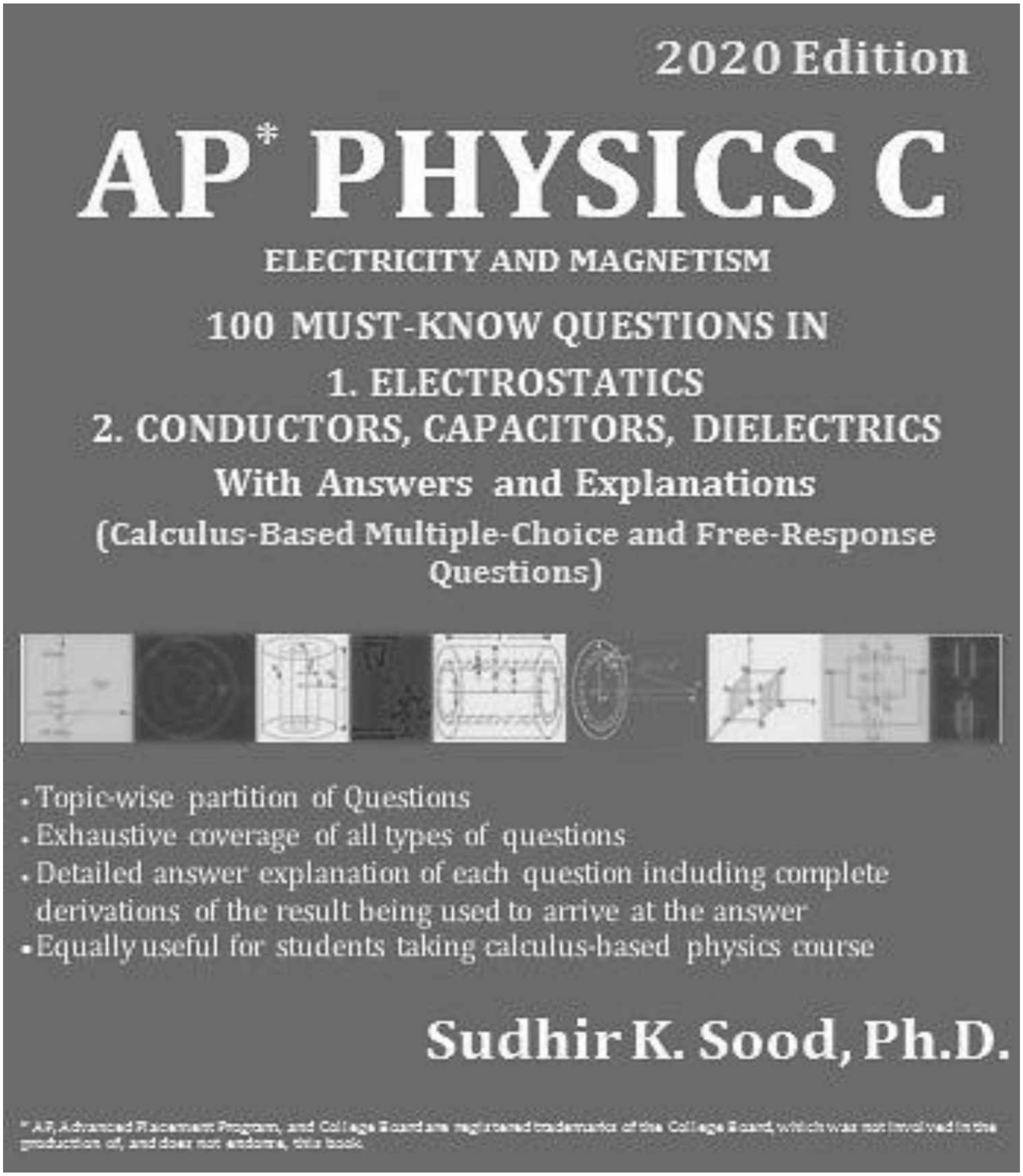
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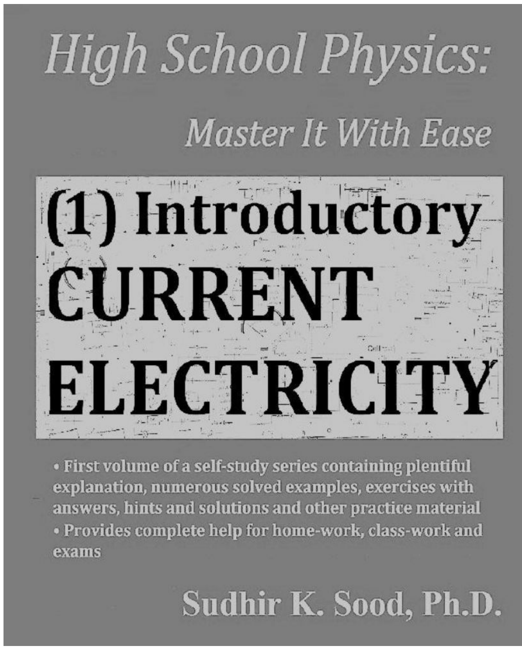
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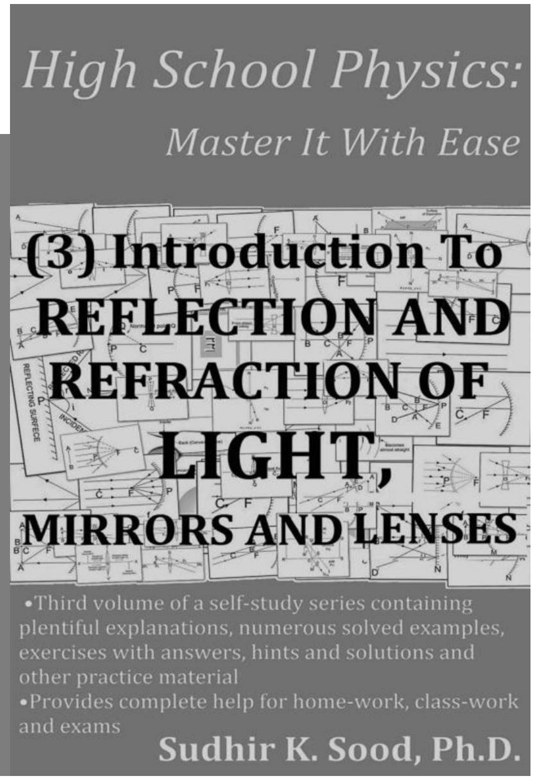
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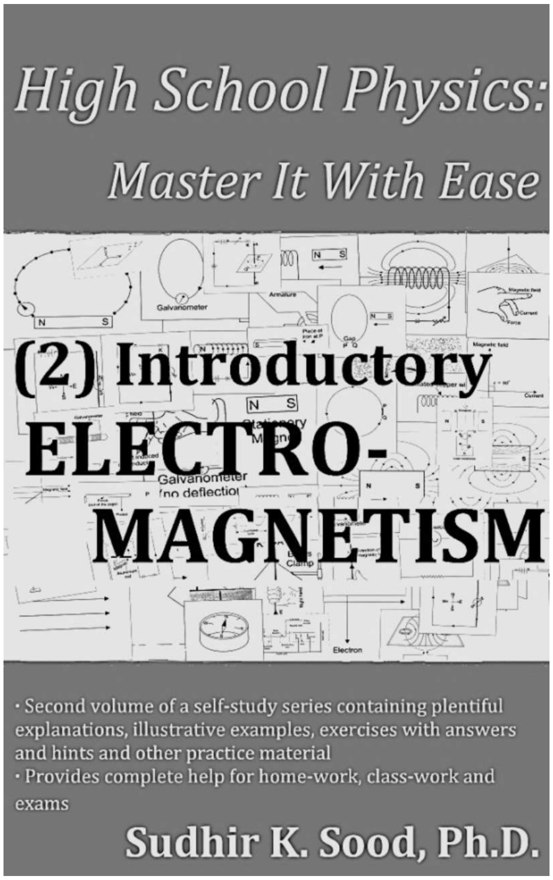
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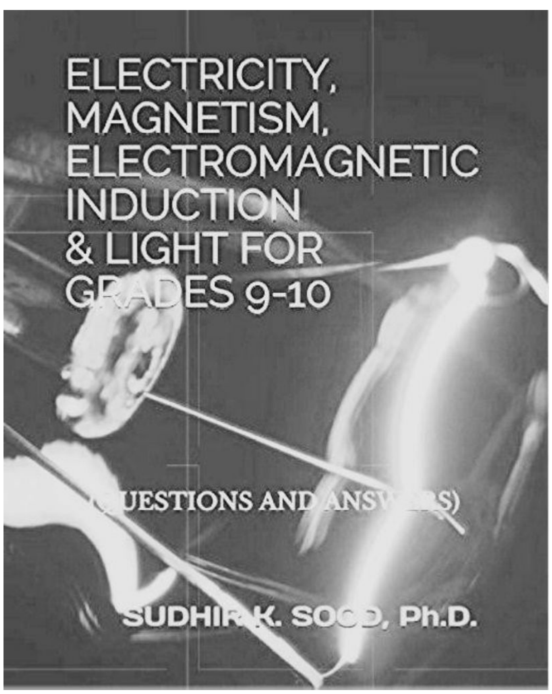
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